PROCEEDINGS
OF THE
NATIONAL ACADEMY OF SCIENCES
Volume 19 October 15, 1933 Number 10

NEON ABSORPTION LINES IN STELLAR SPECTRA

By Donald H. Menzel and Roy K. Marshall
Harvard Observatory and University of Michigan Observatory

Communicated September 14, 1933

The great strength and general occurrence of the "forbidden" emission lines of Ne III and Ne IV in the spectra of nebulae and novae[1] is suggestive of a high cosmic abundance for this supposedly rare gas. It appears extremely unlikely that neon should be plentiful only in these classes of celestial objects or that it should be produced in the process of nova activity. The absorption lines of this atom should appear in normal stellar spectra.

The atom of Ne I[2] has an ionization potential of 21.5 volts. Its resonance lines lie in the far ultra-violet and the only astrophysically observable lines, which lie in the region λλ 5400–7200, have excitation potentials of 16.6 volts. The atom of Ne II[3], ionization potential 40.8 volts, has observable lines in the λ 3700 region, but the excitation potentials are about 27 volts. The visible spectra of higher stages of ionization are still unknown.

Applying the ionization theory as developed by Fowler and Milne,[4] we find that the lines of Ne I and Ne II should come to maximum in stars of respective surface temperatures 12,500° and 22,000°, i.e., near spectral classes B8 and B0. Russell[5] has shown that the elementary theory ought to give a fairly satisfactory representation for temperatures of this magnitude.

A search for the lines of Ne I was made on panchromatic plates taken by Menzel with the three-prism visual spectrograph attached to the 36-inch refractor at Lick Observatory. The strongest line of Ne I, at λ 6402.24, appears to be present as a very faint line in the spectrum of β Orionis (B8p). It is far weaker than the neighboring pair λλ 6347 and 6371 of Si II, which are the most intense lines in the region to the red of λ 5875 He, not excepting Hα. The presence of λ 6402 has been confirmed on three plates and the agreement in wave-length is satisfactory. The line had been measured by Menzel several years ago, when the possibility of attributing it to neon had not been considered. We cannot rule out, however, on the evidence of a single line, the chance that it may be due to some

MULTIPLET	IQ	WAVE-LENGTH LABORATORY	INT. LAB.	WAVE-LENGTH AND INT. STELLAR MARSHALL	STRUVE	IDENT.
$3s^4P\text{-}3p^4P^0$	$2^1/_2\text{–}1^1/_2$	3664.09	9	4.15 (2)[a]		
				4.18 (1)[c]		
	$2^1/_2\text{–}2^1/_2$	3694.22	10	4.28 (4)[a]		
				4.26 (1)[c]		
				4.18 (2)[d]		
	$1^1/_2\text{–}^1/_2$	3709.64	7	9.79 (3)[a]		OIII 9.52 (2)
				9.53 (1)[d]		
	$1^1/_2\text{–}1^1/_2$	3734.94	7	4.33		H 4.370
	$^1/_2\text{–}^1/_2$	3751.26	5	0.16		H 0.154
	$1^1/_2\text{–}2^1/_2$	3766.29	8	0.67		H 0.632
	$^1/_2\text{–}1^1/_2$	3777.16	8	7.22 (3)[a]		OII 7.60 (4)
				7.33 (2)[c]		
				7.11 (2)[d]		
				7.24 (1)[c]		
$3s^2P\text{-}3p^2D^0$	$1^1/_2\text{–}2^1/_2$	3643.89	5	3.74 (2)[a]		
	$1^1/_2\text{–}2^1/_2$	3713.09	10	2.98 (4)[a]		OII 2.75 (7)
				2.83 (4)[d]		H 1.973
				2.79 (7)[c]		OIII 2.48 (2)
	$^1/_2\text{–}1^1/_2$	3727.08	9	7.27 (6)[a]		OII 7.33 (8)
				7.21 (4)[e]		
				7.35 (7)[d]		
				7.38 (6)[c]		
$3p^2P^0\text{-}3d^2D$	$1^1/_2\text{–}1^1/_2$	3800.02	5			
	$^1/_2\text{–}1^1/_2$	3818.44	6	8.27 (1)[a]		
	$1^1/_2\text{–}2^1/_2$	3829.77	7	9.93 (1)[c]		NII 9.80 (3)
$3d^4D\text{-}4f^4D^0$§	$3^1/_2\text{–}3^1/_2$	4219.76	6	0.08 (2)[b]	0.07 (1)[a]	
					9.64 (3)[c]	
	$2^1/_2\text{–}2^1/_2$	4231.60	4		1.70 (2)[b]	
					2.04 (2)[d]	
	$1^1/_2\text{–}2^1/_2$	4250.68	4	0.60 (2)[b]	0.10 (1)[a]	
					0.82 (1)[b]	
$3d^4F\text{-}4f^4G^0$§	$3^1/_2\text{–}4^1/_2$	4391.94	7	2.03 (2)[b]	1.96 (2)[a]	
					1.96 (3)[b]	
					1.96 (2)[d]	
	$2^1/_2\text{–}3^1/_2$	4409.30	7	9.15 (2)[b]	9.27 (3)[b]	
					9.32 (1)[d]	
$3d^4F\text{-}4f^4F^0$§	$1^1/_2\text{–}2^1/_2$	4369.77	5			OII 9.28
	$3^1/_2\text{–}3^1/_2$	4379.50	6	9.14		NIII 9.103 (10)
						OIII 9.55 (3)

[a] 10 Lacertae (O9)
[b] τ Scorpii (B0)
[c] β Cephei (B1)
[d] β Can Maj (B1)
[e] κ Cassiop (B0)
§ Incomplete multiplet

other element. The line could not be found in the spectra of α Canis Majoris, α Virginis, α Leonis or η Ursae Majoris, plates of which were also available.

The lines of Ne II were sought for in spectra obtained by Marshall with the single-prism spectrograph attached to the $37^1/_2$-inch reflector at the Observatory of the University of Michigan. The spectra were measured previous to the present investigation and the lines were merely looked up in the list already prepared. 10 Lacertae (O9) appeared to be the most promising star, because of the large number and sharpness of its lines. A spectrum of κ Cassiopeiae was available, but the lines in this object were much fuzzier and more difficult to see. Two lists of lines by Struve[6] and by Struve and Dunham[7] were also employed. In the table (p. 880) is presented the astrophysical evidence for the occurrence of Ne II in the spectrum of 10 Lacertae.

The respective columns contain the multiplet designations, inner quantum numbers, laboratory wave-lengths and intensities, the stellar wave-lengths and intensities for the various observers and possible identifications other than Ne II. With the exception of λ 3766.29, which may be masked by the neighboring line Hι (stellar intensity 8), all of the lines of the two strongest multiplets are present or accounted for. Several are masked by overlying lines of H or O II. It is interesting to note that λ 3694.22 was the strongest previously unidentified line in the near ultraviolet spectral region of these stars.[8] The stronger lines of several less intense multiplets are probably present. The presence of neon appears to be definitely established. The lines of F II appear to be absent.

Ne II and O II atoms have excitation and ionization potentials of the same order of magnitude. This fact, coupled with the observed similar intensities of the lines, suggests that the two elements have approximately equal abundances. This is roughly substantiated by the intensities of the O III and Ne III forbidden lines in nebulae and novae. Two of the "noble" gases, He and Ne, thus have high abundances in the universe. Cosmically speaking, the term "rare gases" for these elements is a misnomer.

[1] Boyce, Menzel and Payne, these PROCEEDINGS, **19**, 581 (1933).

[2] C. E. Moore, *A Multiplet Table of Astrophysical Interest*, Princeton Observatory (1933).

[3] *Ibid.*, cf. also de Bruin and Bakker, *ZS für Phys.*, **69**, 19 (1931).

[4] Fowler and Milne, *Mon. Not. R. A. S.*, **83**, 403 (1923).

[5] *Astroph. Jour.*, in press.

[6] Struve, *Ibid.*, **74**, 225 (1931).

[7] Struve and Dunham, *Ibid.*, **77**, 321 (1933).

[8] Cf. Marshall, *Ibid.*, **76**, 317 (1932).

TRANSFORMATIONS PRESERVING THE TETRAD EQUATIONS

BY EDWIN B. WILSON

HARVARD SCHOOL OF PUBLIC HEALTH

Communicated September 14, 1933

Recently[1] I gave the form, explicit in terms of the coefficients of an orthogonal transformation, of those linear transformations of a set of k tests which leave Spearman's $\mathbf{g}$ unchanged. An analysis of the same sort finds the explicit form of those transformations which preserve the tetrad equations and thus conserve the possibility of resolution into some general and a set of k specific factors; such a form must include the earlier as a special case. It is merely necessary to transform all the $k + 1$ unit vectors $\mathbf{g}, \mathbf{s}_a, \mathbf{s}_b, \ldots$ as

$$\mathbf{g}' = k_0\mathbf{g} + l_0\mathbf{s}_a + m_0\mathbf{s}_b + \ldots, \quad \mathbf{s}'_a = k_1\mathbf{g} + l_1\mathbf{s}_a + m_1\mathbf{s}_b + \ldots \tag{1}$$

$$k_0^2 + l_0^2 + m_0^2 + \ldots = 1, \quad k_ik_j + l_il_j + m_im_j + \ldots = 0. \tag{2}$$

Then substituting $\mathbf{s}_a = \mathbf{a}\csc\alpha - \mathbf{g}\cot\alpha$, $\mathbf{s}'_a = \mathbf{a}'\csc\alpha' - \mathbf{g}'\cot\alpha'$ with similar expressions for $\mathbf{s}_b, \mathbf{s}_c, \mathbf{s}'_b, \mathbf{s}'_c, \ldots$

$$\mathbf{g}' = \mathbf{g}(k_0 - l_0\cot\alpha - m_0\cot\beta - \ldots) + l_0\csc\alpha\,\mathbf{a} + m_0\csc\beta\,\mathbf{b} + \ldots \tag{3}$$

$$\mathbf{a}'\csc\alpha' = \mathbf{g}[k_0 - l_0\cot\alpha - m_0\cot\beta - \ldots)\cot\alpha' + k_1 - l_1\cot\alpha - m_1\cot\beta - \ldots] + (l_1 + l_0\cot\alpha')\csc\alpha\,\mathbf{a} + (m_1 + m_0\cot\alpha')\csc\beta\,\mathbf{b} + \ldots \tag{4}$$

with similar expressions for $\mathbf{b}'\csc\beta'$, $\mathbf{c}'\csc\gamma', \ldots$; and if the conditions that $\mathbf{g}$ do not appear explicitly in the relations connecting $\mathbf{a}', \mathbf{b}', \mathbf{c}', \ldots$ and $\mathbf{a}, \mathbf{b}, \mathbf{c}, \ldots$ be written, viz.,

$$\begin{aligned}&(k_0 - l_0\cot\alpha - m_0\cot\beta - \ldots)\cot\alpha' + k_1 - l_1\cot\alpha - m_1\cot\beta - \ldots = 0\\ &(k_0 - l_0\cot\alpha - m_0\cot\beta - \ldots)\cot\beta' + k_2 - l_2\cot\alpha - m_2\cot\beta - \ldots = 0,\end{aligned} \tag{5}$$

etc., the values of $\cot\alpha', \cot\beta', \ldots$ are determined from those of $\cot\alpha, \cot\beta, \ldots$ and the assumed coefficients, of which $k(k + 1)/2$ are independent, of an orthogonal transformation on the $k + 1$ unit vectors Then the transformation equations are

$$\begin{aligned}&\mathbf{a}'\csc\alpha' = \mathbf{a}\csc\alpha(l_1 + l_0\cot\alpha') + \mathbf{b}\csc\beta(m_1 + m_0\cot\alpha') + \ldots\\ &\mathbf{b}'\csc\beta' = \mathbf{a}\csc\alpha(l_2 + l_0\cot\alpha') + \mathbf{b}\csc\beta(m_2 + m_0\cot\alpha') + \ldots,\end{aligned} \tag{6}$$

etc. These equations reduce to those which leave $\mathbf{g}$ invariant if $k_0 = 1$ and $l_0 = m_0 = \ldots = k_1 = k_2 = \ldots = 0$.

The part of $\mathbf{g}$ which is determined is the projection $\mathbf{r}_0$ of $\mathbf{g}$ upon the space of $\mathbf{a}, \mathbf{b}, \mathbf{c}, \ldots$ and in the usual notation is

$$\mathbf{r}_0 = r_{tg}\mathbf{t} = \frac{\mathbf{a}\csc\alpha\cot\alpha + \mathbf{b}\csc\beta\cot\beta + \ldots}{1 + Z = 1 + \cot^2\alpha + \cot^2\beta + \ldots} \tag{7}$$

and a similar expression may be written for the known part $\mathbf{r}_0'$ of $\mathbf{g}'$. Both $\mathbf{g}$ and $\mathbf{g}'$ will project upon the space of the tests along the same line so that $\mathbf{r}_0$ and $\mathbf{r}_0'$ will be collinear and $\mathbf{t}'$ will be equal to $\mathbf{t}$ only if the plane $\mathbf{g}' \times \mathbf{g}$ is perpendicular to the space of $\mathbf{a}, \mathbf{b}, \mathbf{c}, \ldots$. The condition for perpendicularity may be expressed as the vanishing of the scalar product of $\mathbf{g}' \times \mathbf{g}$ and the progressive product $\mathbf{a} \times \mathbf{b} \times \mathbf{c} \times \ldots$ of the tests if one follows the notions and notations of a certain type of multi-determinational vector analysis. Now,

$$\mathbf{g}' \times \mathbf{g} = l_0 \csc\alpha\, \mathbf{a} \times \mathbf{g} + m_0 \csc\beta\, \mathbf{b} \times \mathbf{g} + \ldots$$

and the condition $\mathbf{g}' \times \mathbf{g}.\mathbf{a} \times \mathbf{b} \times \mathbf{c} \times \ldots = 0$ is readily shown to be

$$l_0 : m_0 : n_0 \ldots = \cot\alpha : \cot\beta : \cot\gamma : \ldots \tag{8}$$

Indeed that this proportion is sufficient is seen at once from the fact that, if it holds (3) becomes

$$\mathbf{g}' = (k_0 - xZ)\mathbf{g} + x\sqrt{Z(1+Z)}\,\mathbf{t}, \text{ with } k_0^2 + x^2Z = 1,$$

x being a factor of proportionality, and $\mathbf{g}'$ is coplanar with $\mathbf{g}$ and $\mathbf{t}$. Therefore any linear transformation (6) which satisfies (8), and thus involves only a single parameter in addition to the $k(k-1)/2$ which leave $\mathbf{g}$ invariant, will give the same team test $\mathbf{t}$ but different values of $\mathbf{r}_0$, r_{tg} and $\mathbf{g}$. If k_0 be positive, i.e., if the new $\mathbf{g}'$ have a positive component along $\mathbf{g}$ in (1),

$$\mathbf{g}'.\mathbf{t} = \sqrt{\frac{Z}{1+Z}}\,(x + \sqrt{1 - x^2Z}), \qquad 0 < x < \frac{1}{\sqrt{Z(1+Z)}}$$

and as x increases from 0 to $[Z(1+Z)]^{-1/2}$ the value of $\mathbf{g}'.\mathbf{t} = r_{g't}$ increases from r_{gt} to 1. Thus $\mathbf{g}'$ is becoming better and better determined with perfect determination as the limit.[2] If any transformations which do not conserve $\mathbf{g}$ are of significance, it would seem that they should be those which conserve the team $\mathbf{t}$ but render the general factor better determined.

The composition of finite orthogonal transformations is not simple. If we abstract from changes which correspond to mirror reflections, the orthogonal transformation is a rotation and the composition of infinitesimal rotations is simple. Indeed the form of an infinitesimal rotation has for its k^2 coefficients 1 in the main diagonal and infinitesimal quantities which are equal but of opposite sign on opposite sides of the diagonal, namely,

$$\begin{array}{lllll} k_0 = 1 & l_0 & m_0 & n_0 & \ldots \\ k_1 = -l_0 & l_1 = 0 & m_1 & n_1 & \ldots \\ k_2 = m_0 & l_2 = m_1 & m_2 = 1 & n_2 & \ldots \\ k_3 = -n_0 & l_3 = -n_1 & m_3 = -n_2 & n_3 = 1 & \ldots \end{array}$$

which contains explicitly just $k(k + 1)/2$ infinitesimal parameters and the whole transformation may be factored into two, namely,

$$\begin{array}{lllll} k_0 = 1 & l_0 = 0 & m_0 = 0 & n_0 = 0 & \ldots \\ k_1 = 0 & l_1 = 1 & m_1 & n_1 & \ldots \\ k_2 = 0 & -m_1 & m_2 = 1 & n_2 & \ldots \\ k_3 = 0 & -n_1 & -n_2 & n_3 = 1 & \ldots \end{array}$$

and

$$\begin{array}{llll} k_0 = 1 & l_0 & m_0 & n_0 & \ldots \\ -l_0 & 1 & 0 & 0 & \ldots \\ -m_0 & 0 & 1 & 0 & \ldots \\ -n_0 & 0 & 0 & 1 & \ldots \end{array}$$

of which the first contains $k(k - 1)/2$ parameters and does not change $\mathbf{g}$ while the second does change $\mathbf{g}$ and contains k parameters. This may be worked over into a transformation (infinitesimal) of $\mathbf{a}, \mathbf{b}, \mathbf{c}, \ldots$ into $\mathbf{a}', \mathbf{b}', \mathbf{c}', \ldots$. But an infinitesimal transformation may be written down at once in terms of the vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}, \ldots$ and the reciprocal[3] set generally denoted by $\mathbf{a}', \mathbf{b}', \mathbf{c}', \ldots$. If $\mathbf{a}_1, \mathbf{b}_1, \mathbf{c}_1, \ldots$ are then the new vectors (rather that $\mathbf{a}', \mathbf{b}', \mathbf{c}', \ldots$ which are used for the reciprocal set) and $\epsilon, \eta, \zeta, \ldots$ are infinitesimals.

$$\mathbf{a}_1 = (1 + \epsilon)\mathbf{a} - \epsilon\,\mathbf{a}', \quad \mathbf{b}_1 = (1 + \eta)\mathbf{b} - \eta\,\mathbf{b}', \quad \mathbf{c}_1 = (1 + \zeta)\mathbf{c} - \zeta\,\mathbf{c}', \ldots$$

[1] These PROCEEDINGS, **19**, 768–772 (1933). A recent paper by H. T. H. Piaggio, *Brit. J. Psychol.*, **24**, 88–105 (1933), gives an excellent and mathematically elementary summary of some parts of the theory of g with an extensive list of references.

[2] In the limiting case (5) taken with $k_0 - l_0 \cot \alpha - m_0 \cot \beta \ldots = 0$, which must hold in the limit, make $k + 1$ equations linear in the k quantities $\cot \alpha, \cot \beta, \ldots$, which means that the determinant of the coefficients of the orthogonal transformation must vanish, as is impossible. Hence the inference is that one of the quantities $\cot \alpha', \cot \beta', \ldots$ becomes infinite in the limit so that one of the expressions $k_i - l_i \cot \alpha - m_i \cot \beta - \ldots$ need not vanish; in this case one of the transformed tests $\mathbf{a}', \mathbf{b}', \ldots$ becomes identical with $\mathbf{t} = \mathbf{g}$.

[3] The reciprocal set are not unit vectors being defined by the conditions that $\mathbf{a}', \mathbf{b}', \ldots$ are respectively perpendicular to all the vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}, \ldots$ except $\mathbf{a}, \mathbf{b}, \ldots$ and so adjusted in magnitude that $\mathbf{a}'.\mathbf{a} = \mathbf{b}'.\mathbf{b} = \ldots = 1$. If we multiply out a typical tetrad neglecting infinitesimals of higher order $(\mathbf{a} + \epsilon\mathbf{a}') \times (\mathbf{b} + \eta\mathbf{b}').(\mathbf{c} + \zeta\mathbf{c}') \times (\mathbf{d} + \kappa\mathbf{d}') = \mathbf{a} \times \mathbf{b}.\mathbf{c} \times \mathbf{d} + \kappa\mathbf{a} \times \mathbf{b}.\mathbf{c} \times \mathbf{d}' + \zeta\mathbf{a} \times \mathbf{b}.\mathbf{c}' \times d + \ldots$, we have for such products as $\mathbf{a} \times \mathbf{b}.\mathbf{c} \times \mathbf{d}' = \mathbf{a}.\mathbf{c}\ \mathbf{b}.\mathbf{d}' - \mathbf{a}.\mathbf{d}'\mathbf{b}.\mathbf{c}$ nothing but zero because $\mathbf{d}'$ is perpendicular to $\mathbf{b}$ and $\mathbf{a}$. Hence the tetrad product vanishes if that for $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ does. The reason for the coefficient $1 + \epsilon$ to $\mathbf{a}$ in $\mathbf{a}_1$ is to satisfy the condition that $\mathbf{a}_1$ be a unit vector, i.e., $\mathbf{a}_1.\mathbf{a}_1 = 1 + 2\epsilon - 2\epsilon\mathbf{a}.\mathbf{a}' = 1$ by virtue of $\mathbf{a}.\mathbf{a}' = 1$. One may readily show that tetrad formed of four infinitesimal modifications of $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ by additions of multiples of $\mathbf{t}$ must vanish and hence that $\mathbf{a}_1 = x\mathbf{a} + \epsilon\mathbf{t}$, $\mathbf{b}_1 = y\mathbf{b} + \eta\mathbf{t}$, $\mathbf{c}_1 = z\mathbf{c} + \zeta\mathbf{t}$, $\mathbf{d}_1 = w\mathbf{d} + \kappa\mathbf{t}$, where $\epsilon, \eta, \zeta, \kappa$ are infinitesimal and x, y, z, w are 1 except for the infinitesimal adjustment necessary to make $\mathbf{a}_1, \mathbf{b}_1, \mathbf{c}_1, \mathbf{d}_1$ unit vectors, is a transformation which preserves the tetrad equations and contains k parameters.

COLOR CHANGES IN THE ISOLATED SCALE IRIDOCYTES OF THE SQUIRREL FISH, HOLOCENTRUS ASCENSIONIS (OSBECK)[1]

BY DIETRICH C. SMITH

MARINE BIOLOGICAL LABORATORY, WOODS HOLE

Communicated September 5, 1933

Investigation of the color changes in fishes has centered largely about the behavior of the chromatophores, and as a consequence our knowledge of the part played in these changes by those structures called guanophores or iridocytes has not kept pace with what we know about the more active and more accessible melanophores and lipophores. Certain observations which I happened to make upon the iridocytes found in the scales of the common squirrel fish, Holocentrus ascensionis (Osbeck) are of interest in this respect, and may help in throwing additional light on the problem. Foster (1933) has published similar results obtained with certain iridocyte aggregations called iridosomes which lie directly beneath the scales of Fundulus heteroclitus and the observations on the squirrel fish offer an interesting corroboration of those reported by Foster, especially as both investigations were done independently. Before proceeding further I wish to acknowledge my indebtedness to the Bermuda Biological Station, where this work was done in the spring of 1933, and to express my sincere appreciation of the courtesies extended to me by Dr. J. F. G. Wheeler, the Director, during my stay in Bermuda.

The terms iridocyte (Pouchet, 1876) and guanophore (Ballowitz, 1912) have been used more or less interchangeably in the literature and have come to be practically synonymous in meaning. They refer to structures usually found in the deeper layers of the skin which contain guanin crystals either as irregularly formed granules, or as larger and more discrete units in the shape of plates or elongated needle-like bodies. Ginsberg (1929) has described two types of guanophores in the guppy, Lebistes reticulatus, one form presenting a dull white appearance with fine granular guanin, while the other contains large plate-like guanin crystals showing the characteristic iridescence from which the name iridocyte derives its meaning. Odiorne (1933) designates the first type a guanophore, reserving the name iridocyte for the latter. This distinction may be justifiable on other grounds than appearance alone, for certain guanophores apparently have the capacity of altering the distribution of their guanin crystals after the manner of pigment granules in the chromatophores. This was first described by Ballowitz (1913) in Gobius minutus when he says that "die Masse der Guaninkristalle kann sich daher, ähnlich den schwarzen Pigmentkörnchen, ausbreiten und zusammenballen." Jost (1926) from a study of fixed material from Callionymus lyra came to

the same conclusion. According to Odiorne (1933) in Fundulus heteroclitus certain of the guanophores in intimate relationship with certain of the melanophores appear expanded when the fish is kept upon a white background and contracted when kept upon a black background, responses just the reverse of those seen in the associated melanophores. Injection of adrenalin or antuitrin also leads to an expansion of the guanophores and a contraction of the melanophores. Ginsberg (1929) also reports in the embryonic forms of Lebistes reticulatus movements of the guanin granules in the guanophores during the early stages of their development. In all of the cases just mentioned the guanin is present in the form of granules having at least a superficial resemblance to the pigment granules in the chromatophores. Meyer (1931), however, reports in Gobius minutus changes in the shape of guanophores in which the guanin is present in the form of plate-like crystals. Representatives of this species kept for eight or ten days on a white background become milky white in color, the skin losing its previous transparent look. When returned to a black background the whiteness disappears, accompanied by an increase in the iridescence of the guanophores. If excised pieces of the tail fin were examined microscopically, those from fishes kept on a white background show guanophores which are larger in size than those seen in pieces from the tail of an animal kept on black. Meyer ascribes this difference to some sort of an ameboid movement occurring in the guanophore which produced their expansion when the animal was on a white background, thus permitting the guanin plates to spread out over a larger area. On black the guanophores again contracted, this contraction crowding the guanin crystals together to such an extent that they overlapped one another thereby increasing the interference colors, and consequently the iridescence of the cell. Neither Ballowitz, Jost nor Odiorne mention iridescence in connection with those guanophores which they describe as showing a movement of their guanin granules.

Microscopic examination of an isolated scale removed from the trunk of Holocentrus ascensionis (Osbeck), the common squirrel fish, and placed in sea water disclosed in the distal half a region occupied by numerous elongated needle-like structures having a purple-red color and showing some signs of iridescence. In the same region there are many chromatophores but there is no sign of any intimate association between these chromatophores and the other structures present. Since these purple-red bodies have the characteristics of iridocytes rather than guanophores they will be so designated in the following discussion. When the scale is transferred to either $M/10$, NaCl, KCl or $CaCl_2$ this purple color begins to change, a process which after passing through a series of definite variations, eventually leads to the disappearance of all color from the iridocytes. Scale iridocytes of this type are a rather rare occurrence and the ad-

vantage in having such material for further investigation into the nature and function of these structures is obvious. Figure 1, a photograph of a small portion of a scale removed from the trunk of the squirrel fish and placed in sea water, shows the distribution and relative numbers of its iridocytes and chromatophores. The dark elongated needle-like bodies, closely packed together throughout the figure are the purple-red iridocytes. The small irregularly rounded bodies are semi-expanded erythrophores and xanthophores, and the large branched body near the center is an ex-

FIGURE 1

A portion of an isolated scale removed from the trunk of Holocentrus ascensionis and immersed in sea water. The dark elongated bodies are the purple iridocytes. The more conspicuous round objects are slightly expanded erythrophores and xanthophores. Near the center is an expanded melanophore.

panded melanophore. The iridocytes are obviously numerous, so much so that at times the determination of their individual outlines is difficult. In figure 2 the same spot on the same scale is shown after a ten-minute immersion in $M/10$ KCl. As a consequence of the shift from sea water all traces of the iridocytes have vanished, their purple color has disappeared completely and they have become virtually transparent, their outlines failing to register on the photographic plate. In the same figure the erythrophores, xanthophores and melanophores are well contracted, the latter being the large, round object in the center. A similar color loss of the

iridocytes was observed when the isolated scales were immersed in either $M/10$ NaCl or $CaCl_2$. No other solutions were tried. In sea water the characteristic purple-red color of the iridocytes was maintained for at least four hours.

When an isolated scale was subjected to this change from sea water to any one of the mentioned salt solutions, the first iridocytes to show any variations in color were those lying along the proximal border. These changed immediately from a purple to a deep sky blue, which faded rapidly

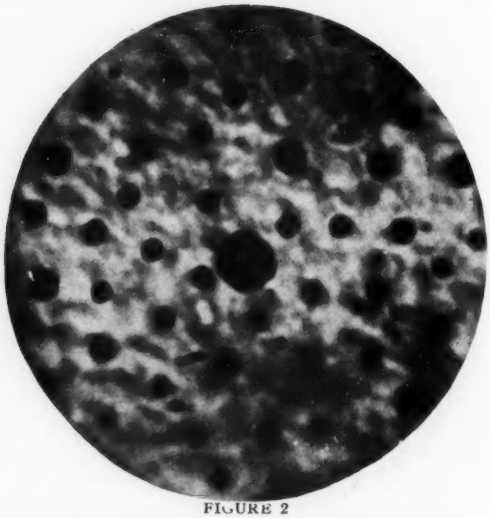

FIGURE 2

A portion of an isolated scale removed from the trunk of Holocentrus ascensionis and immersed in $M/10$ KCl. The same region in the same scale as shown in figure 1. The iridocytes are colorless and do not show in the figure. The small round dots are contracted erythrophores and xanthophores. The large round dot in the center is a contracted melanophore.

to a pale blue and then became pale yellow. The yellow in turn gradually disappeared and the iridocyte was left colorless and transparent. Its outlines, however, were still discernible due probably to the refraction of the light about its edges. It is interesting to note that a green phase was not included among those assumed by the iridocyte during this process. The whole series of color display in any one individual iridocyte was run through in one to two minutes, but since it required a longer time than this for the salt solution to penetrate from its point of entrance along the proximal border of the pigmented area to the distal tissues of the scale,

the iridocytes of the scale showed a striking array of colors during the time the salt solution was working its way toward the distal border. Such a scale in which the penetration was not yet complete showed a proximal band of colorless iridocytes, its width depending on the time elapsing since the transfer from sea water, followed by a pale yellow band which merged gradually into one of pale blue, and finally a deeper blue area shading off into the purple region, the area presumably not yet reached by the salt solution. As the time since the transfer from sea water lengthened, the colorless region gradually widened, its advance being marked by the pushing of the yellow and blue bands toward the distal border, and a consequent narrowing of the purple band until finally it disappeared, a similar fate in turn overtaking the blue and yellow bands eventually leaving all of the iridocytes without a trace of color. From the time the first proximal iridocyte began to change until the last distal one was colorless amounted to from ten to fifteen minutes. Accompanying the changes in the iridocytes and keeping close step with them were the effects of the salt solutions upon the chromatophores. If the scale, for example, were immersed in $M/10$ KCl, a salt which contracts all the pigment cells, the contraction did not appear in the xanthophores and erythrophores until the surrounding iridocytes were pale yellow, the contraction of the lipophores keeping pace with the distally directed progress of the yellow band. In the case of the melanophores this was not true. These cells were not completely contracted until the surrounding iridocytes had been colorless for about three or four minutes.

The loss of color of the iridocytes was not in any way permanent, but was easily reëstablished by removing the scale from either the $M/10$ NaCl, KCl or $CaCl_2$ and returning it to sea water. Such treatment led to the eventual return of the purple color in all the iridocytes. As before, the change set in at the proximal edge of the pigment region, the order of colors being the reverse of that seen before. The colorless proximal iridocytes first became yellow, then blue and finally purple, these changes producing yellow, blue and purple bands in the pigmented area as the sea water penetrated deeper into the scale. These bands, as before, worked their way distally in the order mentioned until all of the iridocytes were purple. If the scale were then returned to any one of the mentioned salt solutions the iridocytes again became colorless. This may be repeated for several hours with no lessening of the response.

While no extensive experiments were undertaken to ascertain the nature of the reaction an attempt was made to find out what effect an increase in the concentration of NaCl had upon the response. These results are best shown in table 1.

From these data it is apparent that as the concentration of NaCl increases, the amount of color change or color loss shown by the iridocyte

is decreased until a point is reached where the color shown in sea water and NaCl is the same. The first sign of this is apparent at $2M/10$ NaCl. At this point the iridocytes in every instance went from a purple to a blue color, but failed to proceed beyond this during the time of observation. An increase in this tendency was noticeable at $3M/10$ NaCl where in one case (Experiment No. 2) the iridocytes showed a change from purple to blue within the first five minutes after removal from sea water, but later, in the course of three and one-half hours, had gone back again to purple.

TABLE 1

SHOWING THE RELATIONSHIP BETWEEN THE COLOR OF THE IRIDOCYTES AND THE CONCENTRATION OF THE NaCl SOLUTION IN WHICH THEY ARE IMMERSED IN COMPARISON WITH THEIR COLOR IN SEA WATER. "NONE" INDICATES THAT THE IRIDOCYTES ARE COLORLESS

EXPERIMENT	TIME	$5M/10$	$4M/10$	$3M/10$	$2M/10$	$M/10$	$5/100$	$M/100$	SEA WATER
	5′	purple				none	none	pale blue	
No. 1	1 hr.	purple				none	none	none	
	4½ hr.	purple				pale blue	none	none	
	5′	purple	bright blue	blue	pale blue	none			purple
No. 2	3½ hr.	purple	purple	purple	bright blue	pale blue			purple
	5′	purple	purple	purple	bright blue	none			purple
No. 3	1 hr.	purple	purple	purple	bright blue	none			purple
	3 hr.	purple	purple	purple	bright blue	none			purple
	5′	purple	purple	bright blue	pale blue	none			
No. 4	1½ hr.	purple	purple	blue	pale blue	none			

In the other two cases at this concentration (Experiments No. 3 and No. 4) one showed no change of any sort in the purple color even after three hours and in the other the iridocytes went from purple to blue and maintained this latter color for at least one and one-half hours. At $4M/10$ and $5M/10$ NaCl, however, the iridocytes remained purple. In all cases at any concentration between $M/10$ and $M/100$ NaCl the change was complete and the iridocytes became colorless, although as is shown in Experiments No. 1 and No. 2 after four and one-half hours and three and one-half hours, respectively, the colorless condition was replaced by a pale blue. The

controls in sea water all maintained their purple color. To summarize the salient features: In $4M/10$ to $5M/10$ NaCl the iridocytes maintain the purple color characteristic of immersion in sea water; between $3M/10$ and $2M/10$ NaCl they show a tendency to change from purple to blue but fail to go further; at concentrations of $M/10$ or less they become completely colorless.

Observations failed to yield any positive information in regard to the mechanism by which this color change in the iridocytes was brought about. Each guanophore contained what looked to be two or three needle-like crystals, these crystals giving the structure its color. If, in a scale taken from sea water and immersed in $M/10$ NaCl, the crystals were watched while they changed from purple to blue and from blue to yellow to become finally colorless, no evidence of any sort of variation other than this alteration in color could be observed in the crystals. No movement of any sort such as a turning or a shifting in their positions could be seen at this time and it was assumed that they remained stationary throughout the process. Occasionally one of the crystals in an iridocyte would change at a rate different from the other so that it was not uncommon to see in an iridocyte a blue crystal side by side with a purple or a yellow one, or a blue crystal persisting in an iridocyte in which the rest of the crystals were colorless. In the majority of the cases, however, the crystals all changed at about the same time, although the occasional exception to this rule would indicate that they react independently of one another.

Mention was made at the beginning of this paper of results obtained by Foster (1933) on Fundulus heteroclitus similar to those just described. According to Foster, in fishes where some of the iridosomes of the trunk have been exposed to the external environment by removing a few of the overlying scales, these iridosomes have a blue or green color, if the animals are kept in sea water. If the fishes are transferred to tap water the iridosomes turn yellow within five minutes. Returned to sea water the iridosomes once more become a dark green or blue, again after five minutes. Furthermore an isolated piece of scale free skin in sea water shows green or blue iridosomes which when the skin is placed in tap water turn yellow and when replaced in sea water return to their former color. He repeated the experiment several times on the same iridosomes always with the same results. He attributed this response to ionic and osmotic changes set up in the exposed skin by the change from sea water to tap water and back again. Ordinarily, of course, this region of the skin is shielded from any such external influence by the epidermis, but when the scales are removed the epidermis is broken and this protection is lost.

The reaction in the iridosomes of Fundulus as described by Foster is undoubtedly identical in nature with the response observed in the iridocytes of isolated Holocentrus scales. In the squirrel fish the removal of

the scales of course breaks their epidermal covering and in so doing furnishes a point of entry through which the solutions in which the scales are placed may penetrate into the tissue spaces surrounding the chromatophores and iridocytes. The scales of this fish are so arranged that when they are removed from the trunk the epidermis is broken near the proximal border of the pigmented area of the scale, which accounts for the responses on the part of the chromatophores and iridocytes first appearing at this point. In view of the results obtained with different concentrations of NaCl the change in color of the iridocytes seems to be conditioned by osmotic rather than ionic stimuli. In solutions hypotonic to sea water the iridocytes lose their color, in hypertonic solutions they do not. In solutions near the isotonic point a shift in color is initiated which if carried to completion would bring the iridocyte to the colorless condition, but which in this instance stops short of this mark. One noteworthy difference between the iridosomes of Fundulus and the iridocytes of Holocentrus is that in the former these bodies change color only, while in the latter they not only change color, but they also lose it completely. This may be attributable to some slight difference in the structure or the arrangement of the guanin layers composing the guanin crystals in these two types of cells. Foster's suggestion that the changes observed are due possibly to a progressive thickening and thinning of the crystal laminae which make up the guanin crystal in such a manner that the wave-length of light reflected from their surfaces is altered, seems to be without any serious objection especially as in both Fundulus and Holocentrus no movement such as rotation or rearrangement of the guanin crystals was seen. The iridosomes of Fundulus upon which Foster worked are not to be confused with the guanophores discussed by Odiorne (1933).

[1] Contribution from the Bermuda Biological Station.

Ballowitz, E., "Ueber schwarz-rote und sternförmige Farbzellenkombination in der Haut von Gobiiden. Ein weiterer Beitrag zur Kenntnis der Chromatophoren und Chromatophoren—Vereinigungen bei Knochenfischen," *Zeit. wiss. Zool.*, Bd. **106**, S. 527–593 (1913).

Foster, K. W., "Color Changes in Fundulus with Special Reference to the Color Changes of the Iridosomes," *Proc. Nat. Acad. Sci.*, **19**, 535–540 (1933).

Ginsberg, J., "Beiträge zur Kenntnis der Guaninophoren und Melanophoren," *Zool. Jahrb. Abt. Anat.*, Bd. **51**, S. 227–260 (1929).

Jost, F., "Die Farbzellen und Farbzellvereinigungen in der Haut der Nordseefisches Callionymus lyra L.," *Zeit. mikro-anat. Forsch.*, Bd. **7**, S. 461–502 (1926).

Meyer, Eva, "Versuche über den Farbwechsel von Gobius und Pleuronectes," *Zool. Jahrb. Abt. allg. Zool. Physiol. Tiere*, Bd. **49**, S. 231–270 (1931).

Odiorne, J. M., "The Occurrence of Guanophores in Fundulus," *Proc. Nat. Acad. Sci.*, **19**, 750–754 (1933).

Pouchet, G., "Des changements de coloration sous l'enfluence des nerfs," *Jour. Anat. Physiol.*, **12**, 113–165 (1876).

MULTIPLE TESTS IN THE FORAMINIFERA

BY EARL H. MYERS

SCRIPPS INSTITUTION OF OCEANOGRAPHY, UNIVERSITY OF CALIFORNIA, LA JOLLA, CALIFORNIA

Communicated August 20, 1933

In marine bottom-deposits free from excessive mechanical injury it is not unusual to find two or more tests of the same species of foraminifera cemented together in such a manner that the apertures are approximately opposed to each other. Multiple tests of this kind have been reported in eight of the families listed by Cushman (1928). In his recent contribution, "A Manual of the Foraminifera," J. J. Galloway (1933) regards these multiple tests as "identical twins, produced from the same egg," and states that "conjugation of individuals is unknown for the foraminifera."

Conjugation in the sense that this word is used by protozoölogists to describe partial syngamy in the Ciliata does not occur in the foraminifera and has never been seriously considered. However, the union of two or more individuals for the purpose of reproduction does occur. To this method of reproduction Fritz Schaudinn applied the term "plastogamy." I am not prepared to say that plastogamy occurs in all of the species in which multiple tests have been described. However, plastogamy is a normal method of reproduction in several families of the foraminifera. The photographs of living organisms figured in the present paper attest this fact and indicate the frequency with which one may encounter plastogamic groups in cultures maintained at temperatures comparable to those found in shallow marine basins during the summer months.

As early as 1880, Möbius discussed the possibility of sexual conjugation in the foraminifera, basing his discussion on four sets of paired tests, out of sixteen specimens taken from the intestine of a holothurian. Brady, in the Challenger report (1884), made several references to these paired tests and discussed at some length the possibility that they might be explained in the same manner as those found in the monothalamous Testacea where paired tests are the result of reproduction by budding.

In 1895, Schaudinn described plastogamy in *Patellina corrugata* Williamson and *Discorbis globularis* d'Orbigny. He observed no evidence of an exchange of nuclear material between the several individuals, and he supposed the resultant brood of young to possess the same type of shell as the parents from which they were derived. Schaudinn's paper is the only published description of this method of reproduction based on observations of living foraminifera.

Earland in 1905, in his paper on the foraminifera of the Bognor Shore-Sands, Sussex, reported that he had in his collections and notes records

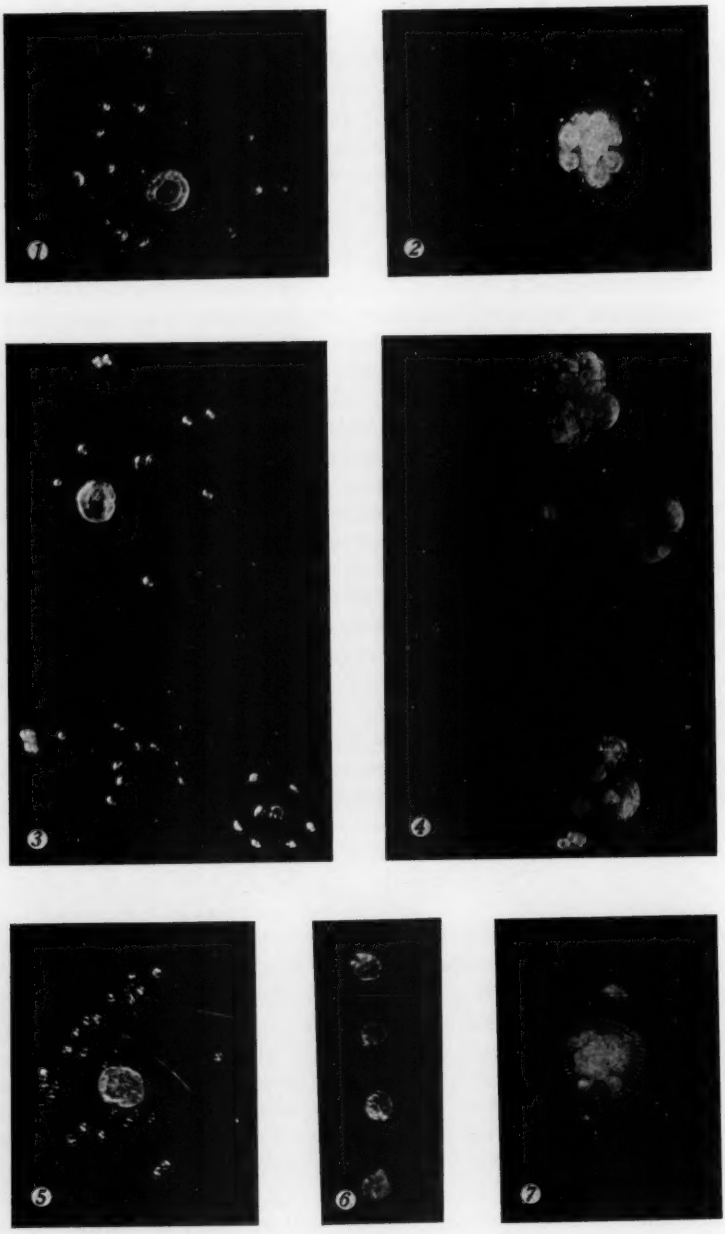

PLATE 1. (Description on opposite page.)

of twelve genera in which he had found "plastogamic pairs," *Peneroplis, Textularia, Verneuilina, Bulimina, Lagena, Marginulina, Cristellaria, Patellina, Discorbina, Rotalia, Nonionina* and *Polystomella.*

Heron-Allen published in 1915 several series of figures of selected paired tests of polythalamous foraminifera, believing that he had demonstrated a new method of reproduction in this group, namely, reproduction by budding. He contended that the observed wide variation in size of the individuals of a pair bore evidence of this method of reproduction. Plate 1, figure 7, is a photograph of living organisms, *Patellina corrugata,* and indicates the degree of variation which may be encountered in individuals of a plastogamic group. It is apparent that these individuals have associated for the purpose of reproduction, since the three tests are enclosed within an envelope of agglutinated debris, and the protoplasmic content of the several tests has given rise to a number of nucleated amoebulae. Heron-Allen's idea of budding is essentially the same as that published by Brady and is based on circumstantial evidence provided by a few empty shells. Unfortunately, his interpretation has gained wide acceptance over Schaudinn's observations which were made on living organisms.

The frequency with which these multiple tests are encountered in certain species eliminates the possibility of "chance association." In many species the tests are so firmly held together by a deposit of lime that when one attempts to separate the members of a pair the ventral side of one of the shells usually breaks away. In a species such as *Discorbis patelliformis* Brady, many chambers intervene between the ventral side of the test and the embryonic chamber. In a plastogamic pair the initial chambers of the

---

EXPLANATION OF PLATE 1

(All figures × 47)

Figure 1. *Spirillina vivipara,* asexual reproduction.

Figure 2. *Patellina corrugata,* asexual reproduction, the young nearly ready to leave the parent test.

Figure 3. *Spirillina vivipara,* the twelve asexual young in figure 1 have associated in six pairs: six young have been produced from each of three of the plastogamic groups.

Figure 4. *Patellina corrugata,* three plastogamic groups of individuals. In the group of five the dark area in the center of each spherule indicates the position of the nucleus. The group of three have fused so recently that the white granular protoplasm has not escaped from the respective tests. In the remaining group most of the protoplasm has escaped. Figures 2 and 4 indicate the relative size of asexual and plastogamic individuals.

Figure 5. *Discorbis patelliformis,* asexual reproduction: the usual number of asexually produced young is about forty.

Figure 6. *Discorbis patelliformis,* plastogamic pairs: the tests remain free to move, by means of pseudopodia, over the surface of the culture cell.

Figure 7. *Patellina corrugata,* a plastogamic group of three individuals, one large and two small, demonstrating the variation in size of individuals which may unite.

two individuals are situated at opposite poles of the major axis. If these multiple tests were formed as suggested by Brady and Heron-Allen, it would be necessary for the two shells to separate completely along the line of fusion before new chambers could be added. This method of growth does not occur, for when a foraminifer becomes permanently attached to a substrate by a deposit of lime, subsequent chambers are added spirally or cyclically about the free test in a horizontal plane, or they may conform to the configuration of the substrate.

Plastogamy, as I have said, constitutes a very important method of reproduction among several widely separated groups of foraminifera, and among those species in which it occurs it may replace reproduction by zoöspores. Earland has by no means exhausted the list of genera in which plastogamy takes place, for in many species the tests are not held together by a permanent cement and become separated in handling. Mal-formed tests, the result of regeneration, are common and one occasionally finds two or more tests of the same or of different species, not plastogamic groups, cemented together. Until we have definite cytological evidence of the occurrence of plastogamy within a given species, the only indication we have as to whether or not this method of reproduction occurs within that species is the frequency with which multiple tests are encountered. One must not be misled by a few questionable specimens.

Schaudinn, in his work on *Patellina corrugata*, selected what is probably the most suitable species for the study of this method of reproduction and, in view of the abundant material at his disposal, it is strange that he should have completely overlooked the mitotic division of nuclei. Very little is known concerning nuclear division in the polythalamous foraminifera. Lister and Schaudinn in "*Polystomella*" (*Elphidium*), and Hofker in *Eponides*, observed what they believed to be a primitive type of mitosis. I have at this time a number of non-decalcified total mounts of *Patellina corrugata* which show large, clear-cut, mitotic figures. This material is especially favorable to work with because it is possible to select from cultures any stage of development or nuclear condition desired. In a later paper on the life history of *Patellina corrugata* I shall publish serial photographs of living organisms showing all phases of plastogamy in this species, and shall include figures and descriptions of the cytological phenomena involved.

The present paper is confined to a brief description of reproduction in *Patellina corrugata*, *Spirillina vivipara* and *Discorbis patelliformis*.

Patellina corrugata Williamson.—Asexual reproduction of *Patellina corrugata* closely parallels that described for "*Polystomella*" (*Elphidium*) by Lister. When an individual reaches the reproductive stage, a protective covering of agglutinated debris consisting largely of diatom frustules encases the test and seals it fast to the substrate. The white granular

protoplasmic content of the test flows into the sub-umbilical depression. This mass of cytoplasm contains about twelve areas which, in reflected light, appear dark in contrast to the surrounding granular cytoplasm and indicate the position of the nuclei. The cytoplasm then becomes concentrated about the nuclei and the differentiation of approximately the same number of young as there are nuclei follows. Before the young *Patellina* break free of the enveloping covering of debris they are provided with a well-formed shell and have a spiral chamber of about one and one-half whorls.

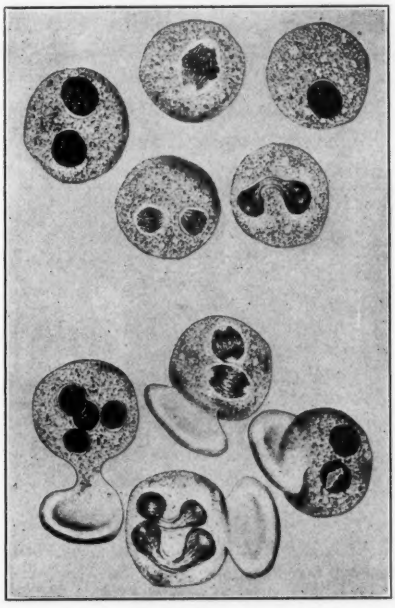

FIGURE 1

Patellina corrugata, mitotic division. A lateral lobe covered by pseudochitin is developed between the first and second divisions. × 470.

When these asexually-produced young attain about two-thirds the diameter of the parent they may associate with individuals of the same or different parents in groups of two or more. I have observed as many as nine. The transparent ectoplasms of the several individuals coalesce to form a common mass about which the tests arrange themselves in a more or less symmetrical manner, so that groups which contain the same number of individuals frequently present the same arrangement. Substrate debris is drawn over the group of tests by pseudopodia to form a protective covering as in asexual reproduction. Plastogamy results in the simultaneous activation of the several individuals and subsequent developmental changes occur at about the same time in the respective tests of a group. The white granular protoplasm in each test escapes into the subumbilical region to form a large opalescent blob; several areas which appear dark in contrast to the surrounding cytoplasm, as may be seen in plate 1, figure 4, indicate the position of nuclei. This blob divides into about four spherical amoebulae, each with a central nucleus. These amoebulae, presumably of different parentage within the group, then unite in pairs to form larger spheres. At this stage there are about twice as

many protoplasmic masses as there are plastogamic tests in the group. A lateral lobe develops on each sphere, the white granular protoplasm gradually clears, and boundaries become indistinct, so that it is difficult to follow subsequent changes. The first indication of calcium deposition in the formation of the test is a curved white line that reflects the light. Later, a spiral line of bright points appears, and at the end of the third day the newly formed young have much the same appearance as those described under asexual reproduction. Plate 1, figure 4, shows three plastogamic groups, one group of five, one of three and a third of two, the last group distorted by an invader. The group of five tests gave rise to ten young, the group of three to six. This ratio of two to one, however, does not always hold.

Spirillina vivapara Ehrenberg.—In *Spirillina vivipara*, the usual number of asexually produced young is also about twelve (plate 1, Fig. 1). Plastogamic groups of two is the rule in this species. Plate 1, figure 2, is a later view of the same group of organisms as that seen in figure 1. These photographs were taken three days apart. The twelve asexual young from the one parent associated to form six groups of two each; each plastogamic pair produced about six young. These photographs are of living material and are typical for this species.

FIGURE 2

Discorbis patelliformis, cross-section through a plastogamic pair. The two tests are completely filled by tri-flagellated organisms. × 470.

Discorbis patelliformis Brady.—In asexual reproduction *Discorbis patelliformis* gives rise to about forty young (plate 1, Fig. 5). Again small individuals of asexual origin form in plastogamic groups of from two to four. Frequently there is a wide variation in the size of these associated tests. I have many times observed all steps in the fusion of these individuals and my observations confirm those of Schaudinn on *Discorbis globularis*. In cultures it was possible to demonstrate very definitely the origin and fate of the tests which formed these groups. I have examined hundreds of multiple tests of this species and prepared many serial sections. In *Discorbis patelliformis* plastogamy results in the production of tri-flagellated organisms, instead of the completely differentiated young as described in *Patellina* and *Spirillina*. Text-figure 2 is a section through a plastogamic pair;

it shows that the two tests are completely filled with these organisms. These flagellates occur in multiple tests directly removed from the sea, and in cultures maintained in the laboratory. I have not yet been able to demonstrate satisfactorily the fate of these flagellates. The majority of asexually produced individuals ultimately form plastogamic groups during the summer months. Schaudinn briefly described broods of young in *Discorbis globularis* resulting from multiple tests, and I have observed this in *Discorbis patelliformis* at rare intervals.

The observations upon which this paper is based have been repeated for three successive seasons. An account of the cytological phenomena involved is being prepared for publication. Meanwhile this announcement is submitted in order that those interested in the foraminifera will no longer have to speculate upon the significance of these multiple tests.

Brady, H. B., Report of the foraminifera dredged by the H.M.S. CHALLENGER during the years 1873–76. Voyage of H.M.S. CHALLENGER, 1873–76, *Zoölogy*, **9** (1884).

Cushman, J. A., "Foraminifera, Their Classification and Economic Use," *Cushman Lab. for Foram. Research*, Spec. Publ. No. 1, Sharon, Mass. (1928).

Earland, Arthur A., "The Foraminifera of the Shore-Sand at Bognor, Sussex," *J. M. C.*, Series 2, No. **57** (1905).

Galloway, J. J., *Manual of Foraminifera*, Principia Press, Inc., Bloomington, Indiana (1933).

Heron-Allen, Edward, "Contribution to the Study of the Bionomics and Reproductive Processes of the Foraminifera," *Roy. Soc., Phil. Trans.*, **B206**, 227–279 (1915).

Hofker, J., "Foraminifera of the SIBOGA Expedition," Pt. 1, *SIBOGA Expeditie*, **4**, Leiden (1927).

Lister, J. J., "Contribution to the Life History of the Foraminifera," *Roy. Soc., Phil. Trans.*, **B186**, 401–453, pls. 6–9 (1895).

Möbius, K., "Foraminiferen von Mauritius. Beiträge zur Meeresfauna der Insel Mauritius und der Seychellen, bearbeitet von K. Möbius, F. Richter und E. von Martens," 4 to 22 pp., Berlin (1880).

Schaudinn, Fritz, "Ueber Plastogamie bei Foraminifera," *Sitz. Berichte Ges. naturf. Freunde*, No. 10, p. 170, Berlin (1895).

AN EXTREMAL PROBLEM IN ANALYTIC FUNCTIONS

BY J. L. WALSH

DEPARTMENT OF MATHEMATICS, HARVARD UNIVERSITY

Communicated September 14, 1933

In the plane of the complex variable z let C be the circumference $|z| = 1$, let the weight or norm function $n(z)$ be positive and continuous on C, and let the function $f(z)$ be integrable (Lebesgue) together with its square on C. To find the function $F(z)$ of class C_2 (i.e., analytic interior to C, with boundary values almost everywhere on C which are integrable and have an integrable square), such that

$$\int_C n(z)\,|f(z) - F(z)|^2\,|dz| \tag{1}$$

is least.

Denote by $g(x,y)$ the function harmonic interior to C, continuous in the corresponding closed region, assuming the values $\frac{1}{2}\log n(z)$ on C, and denote by $h(x,y)$ a function conjugate to $g(x,y)$ interior to C. The function $N(z) = e^{g+ih}$ (introduced in a somewhat different connection by Szegö) is analytic and uniformly limited interior to C, and hence (Fatou) has boundary values almost everywhere on C; the same is true of $1/N(z)$. These boundary values of $N(z)$ on C form a measurable function, uniformly limited. For these boundary values we can write $|N^2(z)| = n(z)$, so the integral (1) can be expressed

$$\int_C n(z)\,|f(z) - F(z)|^2\,|dz| = \int_C |N(z)f(z) - N(z)F(z)|^2\,|dz|. \tag{2}$$

The problem of determining $F(z)$ has thus been reduced to the problem of minimizing an integral where the norm function is unity. The function $N(z)f(z)$ is integrable together with its square on C. The function $N(z)$-$F(z)$ is subjected to the same restrictions as was $F(z)$, namely, that of being of class C_2. For if $F(z)$ is of class C_2, so also is $N(z)F(z)$, and conversely.

If $f_1(z)$ is integrable together with its square on C, the unique function $F_1(z)$ of class C_2 such that

$$\int_C |f_1(z) - F_1(z)|^2\,|dz| \tag{3}$$

is least, is given[1] by

$$F_1(z) = \frac{1}{2\pi i}\int_C \frac{f_1(t)dt}{t-z}, \quad |z| < 1.$$

If we apply this result to the integral in the right-hand member of (2), we find the unique minimizing function

$$F(z) = \frac{1}{2\pi i N(z)} \int_C \frac{N(t) f(t) dt}{t - z}, \quad |z| < 1 \tag{4}$$

and this is a complete solution of our problem.

A more general problem is that of minimizing the integral (1) where we require now that $F(z)$ shall not merely be of class C_2 but shall also satisfy certain auxiliary conditions interior to C of the form

$$F(\beta_k) = \gamma_k, \quad k = 1,2,\ldots\nu, \quad |\beta_k| < 1.$$

As a matter of convention, we admit here points β_k counted multiply; for a point β_k of multiplicity μ, the corresponding μ equations are to be replaced by equations prescribing $F(\beta_k)$, $F'(\beta_k)$, ..., $F^{(\mu-1)}(\beta_k)$.

Let $p(z)$ be a polynomial of degree $\nu - 1$ which satisfies these auxiliary conditions. We are concerned with the integral

$$\int_C n(z) |f(z) - F(z)|^2 |dz| = \int_C \left| N(z) \frac{f(z) - p(z)}{\Pi(z)} - N(z) \frac{F(z) - p(z)}{\Pi(z)} \right|^2 |dz|, \tag{5}$$

where

$$\Pi(z) = \prod_{k=1}^{\nu} \left(\frac{z - \beta_k}{1 - \bar{\beta}_k z} \right),$$

a function analytic on and within C which has the modulus unity on C.

The integral in the right-hand member of (5) suggests a new minimal problem equivalent to the old. The given function is now taken as

$$f_1(z) = N(z) \frac{f(z) - p(z)}{\Pi(z)},$$

which is integrable together with its square on C, and the function to be determined minimizing the integral is

$$F_1(z) = N(z) \frac{F(z) - p(z)}{\Pi(z)}. \tag{6}$$

If $F(z)$ (satisfying the auxiliary conditions) is given of class C_2, equation (6) defines a function $F_1(z)$ of class C_2; to be sure, equation (6) does not define $F_1(z)$ in the points β_k, but the natural definition is to be used in those points. Reciprocally, if $F_1(z)$ is given of class C_2, then equation (6) defines a function $F(z)$ of class C_2 which satisfies the auxiliary conditions. If we now identify the second integral of (5) with the integral (3) and solve for $F(z)$, we have the unique solution of our more general problem:

$$F(z) = p(z) + \frac{\Pi(z)}{2\pi i N(z)} \int_C N(t) \frac{f(t) - p(t)}{\Pi(t)(t - z)} dt, \quad |z| < 1.$$

Our method here used has obvious application to the study of the corresponding problems where C is an arbitrary Jordan curve, to auxiliary conditions consisting of an infinity of equations, to other measures of approximation and also to approximation by polynomials. The writer hopes to treat these and more general applications on another occasion.

[1] F. and M. Riesz, *Compt. Rend. du quatrième congrès* (1916) *des math. scand.* (Uppsala, 1920), 27–44.

ON MOMENT FUNCTIONS

BY EINAR HILLE AND J. D. TAMARKIN

YALE UNIVERSITY AND BROWN UNIVERSITY

Communicated August 23, 1933

1. The present note is a continuation of our note "Questions of relative inclusion in the domain of Hausdorff means."[1] It is also connected with our earlier investigations on the summability of Fourier series by Hausdorff means.[2]

We denote by M_0 the class of all functions of the form

$$\mu(z) = \int_0^\infty e^{-zt} d\alpha(t), \ \Re(z) \geqq 0, \tag{1}$$

where $\alpha(t)$ is of bounded variation in $[0, \infty]$, and is normalized by the conventions $\alpha(0) = 0$ and $\alpha(t) = \frac{1}{2}[\alpha(t + 0) + \alpha(t - 0)]$, $0 < t < \infty$. We denote the total variation of $\alpha(t)$ in $[0, \infty]$ by $V[\alpha]$. The functions of M_0 are bounded in $\Re(z) \geqq 0$, continuous at all finite points of $\Re(z) = 0$ and holomorphic in $\Re(z) > 0$. When desirable, we may think of M_0 as a linear metric vector space where the distance between $\mu_1(z)$ and $\mu_2(z)$ is given by

$$(\mu_1, \mu_2) = V[\alpha_1 - \alpha_2] \equiv W[\mu_1 - \mu_2]. \tag{2}$$

The space is complete with respect to this metric. We denote further by M_h, h real, the class of functions $f(z)$ such that $f(z + h) \subset M_0$, and put $\sum_{n=1}^{\infty} M_n = \lim_{n \to \infty} M_n = \mathbf{M}$.

2. Necessary and sufficient conditions that an analytic function $f(z)$ shall belong to M_0 have been given by F. Hausdorff, S. Bernstein and D. V. Widder.[3] None of these conditions is easy to apply in practice. The basic criterion in terms of the boundary values $f(iy)$ is as follows. We suppose $f(z)$ to be holomorphic for $\Re(z) > 0$, bounded in $\Re(z) \geqq 0$, and continuous at all finite points of the boundary. Put

$$A(t) = \frac{1}{2\pi i} \lim_{T \to \infty} \int_{-T}^{T} f(iy)[e^{iyt} - 1]\frac{dy}{y}. \tag{3}$$

A necessary and sufficient condition that $f(z) \subset M_0$ is that

$$V[A] < \infty. \tag{4}$$

This condition is also none too easy to handle in practice, but it has at least the advantage of connecting the problem with the theory of Fourier transforms. The following observation may serve as an illustration of this connection.

Let H_p, $1 \leqq p \leqq 2$, denote the class of functions holomorphic in $\Re(z) > 0$ such that

$$\int_{-\infty}^{\infty} |f(x + iy)|^p \, dy \leqq C = C_f, \quad x > 0. \tag{5}$$

It can be shown that such a function $f(z)$ possesses boundary values $f(iy)$, defined almost everywhere, and that $f(iy) \subset L_p$, where L_p is the class of measurable functions the pth power of which are integrable over $[-\infty, \infty]$. Put

$$F(t) = (2\pi)^{-1/2} \text{ l.i.m.} \int_{-T}^{T} f(iy)e^{-iyt} \, dy, \tag{6}$$

where l.i.m. denotes the uniform limit if $p = 1$, and the limit in the mean of order p if $1 < p \leqq 2$. Then the following theorem holds.

THEOREM 1. *If $f(z) \subset H_p$, $1 \leqq p \leqq 2$, then $f(z) \subset M_h$ for every $h > 0$. A necessary and sufficient condition that $f(z) \subset M_0$ is that $F(t) \subset L_1$,* i.e., *that*

$$\int_{-\infty}^{\infty} |F(t)|dt < \infty. \tag{7}$$

If (7) *holds, then*

$$f(z) = (2\pi)^{-1/2} \int_0^{\infty} e^{-zt} F(-t)dt. \tag{8}$$

The proof follows essentially from the fact that $f(z)$ can be represented by Poisson's integral for $\Re(z) > 0$ when it belongs to H_p. A somewhat less precise result (in the case $p = 1$) is already in the literature.[4]

A number of conditions ensuring the validity of (7) are to be found in the papers quoted in footnote 2. We recall in particular the following conditions which apply directly to the problem on hand.[5]

(A) *$f(iy)$ is the boundary function of an $f(z) \subset H_p$, $1 \leqq p \leqq 2$, $f(iy)$ is absolutely continuous and $f'(iy) \subset L_1$.*

(B) *$f(iy) \subset L_p$, $1 < p \leqq 2$, and*

$$\int_0^1 \omega(h) \, h^{-1/p-1} \, dh < \infty, \tag{9}$$

where

$$\omega(h) = \left\{ \int_{-\infty}^{\infty} |f(iy + ih) - f(iy - ih)| \ldots |^p \, dy \right\}^{1/p}. \quad (10)$$

These criteria as well as Theorem 1 apply only to the case $1 \leqq p \leqq 2$. But there are functions of M_0 which belong to a class H_p with $p > 2$ as well as functions which do not belong to any such class. Examples are given by the functions of Hausdorff's logarithmic scale

$$(1 + z)^{-\alpha}[\log (a_1 + z)]^{-\alpha_1} [\log_2(a_2 + z)]^{-\alpha_2} \ldots [\log_k(a_k + z)]^{-\alpha_k}, \quad (11)$$

where the first non-vanishing exponent is positive and the a's satisfy obvious restrictions. All these functions satisfy the following condition:

(C_1) *$f(iy)$ is absolutely continuous, $f(iy) \longrightarrow 0$ as $|y| \longrightarrow \infty$, $f'(iy) \subset L_1$, and*

$$\int_0^1 \Omega(h)\, h^{-1}\, dh < \infty, \quad (12)$$

where

$$\Omega(h) = \int_{-\infty}^{\infty} |f'(iy + ih) - f'(iy - ih)|\, dy. \quad (13)$$

Condition (C_1) implies the existence of $F(t)$, the Fourier transform of $f(iy)$ defined by (6) with l.i.m. replaced by lim, for every $t \neq 0$, and ensures the integrability of $F(t)$ except possibly over $(-\epsilon, \epsilon)$. We can impose additional conditions on $f(iy)$ which will ensure the integrability of $F(t)$ near $t = 0$, for instance:

(C_2) *$f'(iy)$ is of bounded variation for large values of $|y|$.*

If $f(iy)$ is the boundary function of a function $f(z)$, holomorphic for $\Re(z) > 0$, and if $f(iy)$ satisfies (C_1) and (C_2), then $f(z) \subset M_0$ and formulas (6), (7) and (8) hold. These criteria are satisfied, for instance, by the functions of (11). In all these cases the corresponding function $\alpha(t)$ is not merely of bounded variation but even absolutely continuous, so that $f(z)$ is representable by a Laplace integral.

3. We have seen that $H_p \subset \mathbf{M}$ for $1 \leqq p \leqq 2$, and that $H_p \cdot \mathbf{M}$ is never vacuous for any fixed $p > 2$. In order to obtain more information concerning $H_p \cdot \mathbf{M}$ when $p > 2$, let us introduce a metric in H_p by taking the distance between $f_1(z)$ and $f_2(z)$ equal to $||f_1 - f_2||_p$ where

$$||f(z)||_p = \lim_{x \to 0} \left\{ \int_{-\infty}^{\infty} |f(x + iy)|^p \, dy \right\}^{1/p} = \left\{ \int_{-\infty}^{\infty} |f(iy)|^p \, dy \right\}^{1/p}. \quad (14)$$

H_p then becomes a complete linear metric vector space. The functions of $\mathbf{M}$ are everywhere dense in this space since $af(z)/(z + a) \subset M_\epsilon \cdot H_p$ and is arbitrarily close to $f(z)$ if a is large positive.

If $f(z) \subset H_p$, then the function $A(t)$ defined by formula (3) is continuous. Moreover, $A(t)$ is a continuous functional of $f(z)$ in the sense that $||f_n(z) - f(z)||_p$

$\longrightarrow 0$ implies $A_n(t) \longrightarrow A(t)$ uniformly in t over every fixed finite interval. We shall show that $A(t)$ is normally not of bounded variation in $[0, \omega]$. For this purpose we introduce the two-parameter family of operations

$$U_{n,\omega}[f] = \sum_{k=1}^{[n\omega]} \left| A\left(\frac{k}{n}\right) - A\left(\frac{k-1}{n}\right) \right|. \tag{15}$$

For every fixed n and ω this is a continuous operation defined all over H_p. Using a familiar argument of Banach-Steinhaus-Saks[6] we conclude from this fact that for every fixed $\omega > 0$

$$U_\omega[f] = \overline{\lim_{n \to \infty}} U_{n,\omega}[f] \tag{16}$$

is finite either all over H_p or else merely in a set of the first category with respect to H_p, i.e., a set which is the union of a denumerable number of nowhere dense sets. In order to prove that the second possibility holds it is sufficient to exhibit a single function $f(z)$ such that $U_\omega[f] = \infty$. It suffices to take

$$f(z) = (1+z)^{-1/2} \prod_{n=2}^{\infty} \frac{n \log^2 n - z}{n \log^2 n + z}, \tag{17}$$

which belongs to every H_p with $p > 2$. The corresponding function $A(t)$ is analytic in the right t-half-plane, but oscillates so strongly as $t \longrightarrow 0$ along the real axis that its derivative is not integrable down to zero, i.e., $A(t)$ is not of bounded variation in any interval $[0, \omega]$ so that $U_\omega[F]$ cannot be finite. We have consequently proved[7]

THEOREM 2. *For every fixed $p > 2$ the functions of H_p such that the corresponding function $A(t)$ is of bounded variation in $[0, \omega]$ form a set of the first category in H_p.*

COROLLARY. *For every fixed $p > 2$ the cross-section of H_p and* **M** *is of the first category in H_p.*

Let us also consider the space B of functions $f(z)$ holomorphic and bounded in $\Re(z) > 0$ with the metric

$$||f_1 - f_2|| = \sup_{x > 0} |f_1(z) - f_2(z)|. \tag{18}$$

Let α be fixed, $0 < \alpha \leqq \frac{1}{2}$, and form $F(z) = (1+z)^{-\alpha} f(z)$ where $f(z) \subset B$. This is a function of H_p when $\alpha p > 1$. A slight modification of the preceding argument leads to

THEOREM 3. *For every fixed α, $0 \leqq \alpha \leqq \frac{1}{2}$, the functions $f(z)$ of B such that $(1+z)^{-\alpha} f(z) \subset$* **M**, *form a set of the first category in B.*

On setting $g(z) = f(z)/z$, where $f(z)$ belongs to one of the classes mentioned in Theorems 2 and 3 it is readily found that $g(z+h) \subset H_p$ for each

$p > 1$ and each $h > 0$. Hence $g(z)$ admits of a representation by a Laplace integral

$$g(z) = \int_0^\infty e^{-zt}\beta(t)dt, \ \Re(z) > 0,$$

and $f(z)$ is obtained by integrating by parts. This result (which follows also from the general theory of Laplace representations developed by Bochner, loc. cit.,[4]) appears to be useful in several problems of the theory of Dirichlet series. We postpone the discussion to a future note.

4. We shall now consider analytic functions of moment functions, and start with

THEOREM 4. *Let* $\mu(z) \subset M_0$, $\mu(\infty) = \alpha(+0) = \alpha_0$, *and suppose that* $F(w)$ *is an analytic function, holomorphic at* $w = \alpha_0$. *Then there exists an* η *such that* $F(\mu(z)) \subset M_h$ *for* $h > \eta$.

For the proof we first observe that

$$W[\mu_1(z)\mu_2(z)] \leqq W[\mu_1(z)]\, W[\mu_2(z)]. \tag{19}$$

Consequently, if

$$F(w) = \sum_0^\infty a_n(w - \alpha_0)^n,$$

convergent for $|w - \alpha_0| < \rho$, we can first choose η so that $W[\mu(z + h) - \alpha_0] \leqq \rho - \delta$ for $h > \eta$. We have then

$$W[F(\mu(z + h))] \leqq \sum_0^\infty |a_n|\ (\rho - \delta)^n < \infty,$$

which shows that $F(\mu(z)) \subset M_h$. In particular, $F(\mu(z)) \subset M_0$ if

$$\sum_0^\infty |a_n|(W[\mu - \alpha_0])^n < \infty. \tag{20}$$

The special cases mentioned below in connection with Theorem 5 show that if the series in (20) diverges, $F(\mu(z))$ may or may not belong to M_0.

In this connection we want to call attention to some generalizations of a theorem of Mercer and I. Schur.[8] This theorem is normally stated in terms of limits. An equivalent formulation is that $[1 + c/(1 + z)]^{-1}$ belongs to M_0 for $\Re(c) > -1$. This is a special case of our

THEOREM 5. *If* $\alpha > 0$, *then* $[1 + c(1 + z)^{-\alpha}]^{-1} \subset M_0$ *if and only if* $\Re[(-c)^{1/\alpha}] < 1$. *In general, if* $\mu(z) \subset M_0$, *then* $[1 + c\mu(z)]^{-1} \subset M_0$ *when* $|c|\, W[\mu] < 1$ *and need not belong to* M_0 *in any larger region*.

The special case is proved by direct computation of the corresponding mass function which is expressible in terms of Mittag-Leffler's function E_α. The general case is a consequence of Theorem 4. If $\mu(z) = e^{-z}$, then $W[e^{-z}] = 1$ and $[1 + ce^{-z}]^{-1}$ does not belong to M_0 for any c such that $|c| \geqq 1$, so that the general statement is in a certain sense the best possible of its kind.

There are of course analogues of Theorem 4 for functions of the type $F(\mu_1(z), \mu_2(z), \ldots, \mu_n(z))$.

If $w = \alpha_0$ is a branch point of $F(w)$ we have a much more difficult question. The basic problem in this connection is the study of pure equations of the type

$$w^m - \mu(z) = 0. \tag{21}$$

Since $\mu(z)$ might have zeros whose real parts tend to ∞, $[\mu(z)]^{1/m}$ need not be holomorphic in any right half-plane in which case it of course cannot belong to $\mathbf{M}$. But even if we suppose that $[\mu(z)]^{1/m}$ is holomorphic in a right half-plane, it need not belong to $\mathbf{M}$. In order to construct such a function we may take $\mu(z) = [f(z + \epsilon)]^m$ where $\epsilon > 0$, $f(z) \subset H_p - H_p \cdot \mathbf{M}$ and $2 < p \leqq 2m$. The existence of such functions $f(z)$ follows from Theorem 2. Here $\mu(z) \subset M_0$ by Theorem 1, but $[\mu(z)]^{1/m} \not\subset \mathbf{M}$.

This example shows that it is necessary to impose additional restrictions on $\mu(z)$ in order that its mth root shall belong to $\mathbf{M}$. A sufficient condition is of course that the mth root belongs to a class H_p, $1 \leqq p \leqq 2$. Another sufficient condition is that $\mu(z)$ is representable by a Dirichlet series as is shown by the theorem of Ritt and Ostrowski, in which case the root can also be expanded in such a series.[9] This result is connected with our Theorem 4 which shows that $[\mu(z)]^{1/m} \subset M_h$ for all large h if $\mu(\infty) = \alpha_0 \neq 0$. These conditions are very far apart and apply to entirely different classes of functions. We can throw no light at present on the intermediary cases.

Analogous considerations apply to more general equations of the type

$$w^m + \mu_1(z)w^{m-1} + \ldots + \mu_m(z) = 0.$$

5. In concluding we have to offer one more negative result. In our preceding note[1] we called attention to the prime functions of M_0, viz.,

$$\frac{\bar{a}}{a} \cdot \frac{a - z}{a + z}, \quad \Re(a) > 0. \tag{22}$$

It is natural to ask if an absolutely convergent infinite product of prime functions necessarily belongs to M_0 or even to $\mathbf{M}$. Thus we are led to the consideration of Blaschke products of the form

$$B(z) = \prod_{n=1}^{\infty} \frac{\bar{a}_n}{a_n} \cdot \frac{a_n - z}{\bar{a}_n + z}. \tag{23}$$

The case in which the a's are real positive is particularly interesting. It seems plausible that no such function $B(z)$ can belong to M. We have verified this in a number of special cases, for instance, when $a_n = n^p$, $p > 1$, and when $a_{n+1}/a_n > 1 + \delta$, $\delta > 0$.[10] On the other hand, it is easy to find Blaschke products with complex roots which $\subset M_0$. An example is given by $(e^{-z+1} - 1)/(e^{z+1} - 1)$, the zeros of which are $a_n = 1 \pm 2n\pi i$.

[1] These PROCEEDINGS, **19**, 573–577 (1933).

[2] "On the Summability of Fourier Series. Third Note," these PROCEEDINGS, **16**, 594–598 (1930); "On the Summability of Fourier Series. III," *Math. Ann.*, **108**, 525–577 (1933).

[3] F. Hausdorff, "Summationsmethoden und Momentfolgen. I," *Math. Z.*, **9**, 74–109 (1921); S. Bernstein, "Sur les fonctions absoluments monotones," *Acta Math.*, **52**, 1–66 (1929); D. V. Widder, "Necessary and Sufficient Conditions for the Representation of a Function as a Laplace Integral," *Trans. Amer. Math. Soc.*, **33**, 851–892 (1931).

[4] See S. Bochner, *Vorlesungen über Fouriersche Integrale*, Leipzig, Satz 47, p. 156 (1932), where further references are to be found.

[5] See our *Math. Annalen* paper, loc. cit.[2] Condition (A) is one form of the analogue of theorem A VII, p. 533, the proof of which will be published elsewhere. Condition (B) is Theorem 6.1 on p. 535; for condition (C_1) compare Theorem 7.1, pp. 538–540.

[6] Cf. S. Saks, "On Some Functionals," *Trans. Amer. Math. Soc.*, **35**, 549–556 (1933). Other references are found there.

[7] This proof is modeled upon a similar argument used by F. Bohnenblust and E. Hille in an unpublished investigation concerning the absolute convergence problem for ordinary Dirichlet series.

[8] J. Mercer, "On the Limits of a Real Variant," *Proc. London Math. Soc.* (2) **5**, 206–224 (1907); I. Schur, "Ueber die Äquivalenz der Cesàroschen und Hölderschen Mittelwerte," *Math. Ann.*, **74**, 447–458 (1913).

[9] J. F. Ritt, "Algebraic Combinations of Exponentials," *Trans. Amer. Math. Soc.*, **31**, 654–679 (1929); A. Ostrowski, "Über algebraische Funktionen von Dirichletschen Reihen," *Math. Z.*, **37**, 98–133 (1933).

[10] See also formula (17).

ON THE THEORY OF LAPLACE INTEGRALS

BY EINAR HILLE AND J. D. TAMARKIN

YALE UNIVERSITY AND BROWN UNIVERSITY

Communicated August 24, 1933

1. This is a continuation of our previous note "On moment functions" which will be referred to as M in the sequel.[1]

We denote by $H_p(a)$ the class of analytic functions $f(z)$ holomorphic in $x > a$, $z = x + iy$, such that

$$\int_{-\infty}^{\infty} |f(x + iy)|^p \, dy \leqq C = C_f, \; x > a, \tag{1}$$

where p is fixed $\geqq 1$. Let further $H_{p,\alpha}(a)$ denote the class of functions $f(z)$ such that $z^{-\alpha} f(z) \subset H_p(a)$. Here α is real and $a \geqq 0$.

In M, §3, we observed that every function of $H_p(0)$ is representable by a Laplace-Stieltjes integral

$$f(z) = \int_0^{\infty} e^{-zu} dA_1(u) \tag{2}$$

for $x > 0$. If $1 \leqq p \leqq 2$, the function $A_1(u)$ is absolutely continuous so that we are actually dealing with an ordinary Laplace integral. $A_1(u)$ is still continuous but normally not of bounded variation when $2 < p$. In this case it is more convenient to write

$$f(z) = z\int_0^\infty e^{-zu} A_1(u)\,du, \tag{3}$$

where the integral, taken in the sense of Lebesgue, is absolutely convergent for $x > 0$. This is merely a particular case of representations of the form

$$f(z) = z^\alpha \int_0^\infty e^{-zu} A_\alpha(u)du. \tag{4}$$

Such representations play a fundamental rôle in the theory of Dirichlet series.[2] When α is an integer (4) reduces essentially to a generalized Laplace integral in the sense of Bochner.[3]

Let us denote by $R_\alpha(a)$ the class of functions representable by an integral of type (4) absolutely convergent for $x > a(\geqq 0)$.

THEOREM 1. *If $f(z) \subset R_\alpha(a)$, then it belongs to every $R_\beta(a)$ with $\beta > \alpha$. If $A_\alpha(0) = 0$ and $\beta = \alpha + \gamma$, then*

$$A_\beta(u) = \frac{1}{\Gamma(\gamma)} \int_0^u A_\alpha(v)\,(u - v)^{\gamma-1}dv. \tag{5}$$

If γ is an integer, a γ-fold integration by parts proves the theorem. If γ is not an integer, it is best to resort to the inversion formulas defining $A_\alpha(u)$ and $A_{\alpha+\gamma}(u)$.

THEOREM 2. *If $f(z) \subset H_{p,\alpha}(a)$, where $1 \leqq p \leqq 2$, then $f(z) \subset R_\alpha(a)$.*

This is an immediate consequence of *M*, Theorem 1.

COROLLARY. *Suppose that $f(z)$ is regular for $x > a \geqq 0$ and that $f(z) = O[|z|^\mu]$ uniformly for $x \geqq a + \delta$, however small be $\delta > 0$, then $f(z) \subset R_\alpha(a)$ for every $\alpha > \mu + \frac{1}{2}$.*

In *M*, formula (17), we gave an example of a function which is regular and uniformly $O[|z|^{-1+\varepsilon}]$ for $x \geqq 0$, but which does not belong to any class $R_0(a)$. It follows that the inequality $\alpha > \mu + \frac{1}{2}$ cannot be improved upon for any μ without additional restrictions on $f(z)$.

2. Let us now consider the question of algebraic functions of functions belonging to a class $R_\alpha(a)$. We begin with the existence of the reciprocal.

THEOREM 3. *Suppose that* (i) $f(z) \subset R_\alpha(a)$, (ii) $f(z)$ *tends to unity uniformly in y as $x \longrightarrow \infty$, and* (iii) $f(z) \neq 0$ *for $x \geqq a_1 > a$. Then $[f(z)]^{-\gamma} \subset R_\mu(a_1)$ for $\mu > \frac{1}{2}$, $\gamma > 0$.*

The proof follows from a theorem of Landau,[4] according to which $[f(z)]^{-1} = O[|y|^\varepsilon]$ uniformly in $x \geqq a_1 + \delta$, combined with the corollary of Theorem 2. The assumption that $f(z) \longrightarrow 1$ can be replaced by $z^\beta f(z) \longrightarrow 1$ for some fixed β in which case the inequality becomes $\mu > \frac{1}{2} + \beta\gamma$ instead. We do not know if these limits are the best possible.

THEOREM 4. *Let the functions* $f_\nu(z) \subset R_{\nu\alpha}(a)$, $\nu = 1, 2, \ldots, m$. *If* $w(z)$ *is a root of the equation*

$$w^m + f_1(z)w^{m-1} + \ldots + f_m(z) = 0 \tag{6}$$

which is holomorphic for $x > a_1 \geqq a$, *then* $w(z) \subset R_\beta(a_1)$ *for* $\beta > \alpha + \frac{1}{2}$.

Putting $w = z^\alpha W$ we are led to a new equation whose coefficients belong to $R_0(a)$. The new coefficients being bounded for $x \geqq a_1 + \delta$, the same is true of the roots, and, by the corollary of Theorem 2, any root belongs to $R_{1/2+\epsilon}(a_1)$ if it is holomorphic for $x > a_1 \geqq a$. The bound $\alpha + \frac{1}{2}$ in the inequality cannot be replaced by any smaller quantity if the resulting inequality is to be true for all values of m. Indeed, for every given value of m we can construct an equation, using formula (17) of M, such that $a_1 = a$ and the roots do not belong to $R_\beta(a)$ for any $\beta \leqq \alpha + \frac{1}{2} - \frac{1}{2m}$.

THEOREM 5. *Let the functions* $f_\nu(z) \subset H_{\nu\rho}(a)$, $\nu = 1, 2, \ldots, m$, $1 \leqq p$. *If* $w(z)$ *is a root of equation* (6) *which is holomorphic for* $x > a_1 \geqq a$ *then* $w(z) \subset H_\rho(a_1)$. *Thus* $w(z)$ *is representable by a Laplace-Stieltjes integral of type* (2) *for* $x > a_1$, i.e., *it admits of a representation of the same type as that of the coefficients of the equation.*

The proof follows from the fact that for every fixed z, $|w(z)| \leqq \text{Max } |f_\nu(z)/m|^{1/\nu}$, $\nu = 1, 2, \ldots, m$.

3. It is not without interest to apply our results to Dirichlet series. In spite of the fact that our theorems are obviously modelled upon known results from the theory of Dirichlet series, they throw additional light upon the latter theory. Suppose that

$$f(s) \sim \sum_1^\infty a_n e^{-\lambda_n s}, \quad 0 \leqq \lambda_1 < \lambda_2 < \ldots, \tag{7}$$

and put

$$A(u) = \sum_{\lambda_n < u} a_n. \tag{8}$$

As is usual in the theory of Dirichlet series, we understand (7) to imply that the formal series on the right is summable in some region R of the plane, the generalized sum being an element of the analytic function $f(s)$. It is convenient to assume that R contains a right half-plane.

The rate of growth of an analytic function on vertical lines can be measured by various means of which the μ-function of Lindelöf,[5]

$$\mu(\sigma;\ f) = \overline{\lim_{|t| \to \infty}} \log |f(\sigma + it)| / \log |t|, \tag{9}$$

and the ν-function of F. Carlson,[6]

$$\nu(\sigma;\ f) = \overline{\lim_{\omega \to \infty}} \log \left[\int_{-\omega}^{\omega} |f(\sigma + it)|^2 dt \right] / \log \omega, \tag{10}$$

are the most convenient.

THEOREM 6. *Let $f(s)$ be associated with the series (7) and let $f(s)$ be regular for $\sigma > \eta \geqq 0$. Suppose further that either $\mu(\sigma; f) < \alpha - \frac{1}{2}$ or that $\nu(\sigma; f) < 2\alpha, \alpha > \frac{1}{2}$, for $\sigma > \sigma_\alpha \geqq \eta$. Then*

$$f(s) = s^\alpha \int_0^\infty e^{-su} A_\alpha(u) du \tag{11}$$

for $\sigma > \sigma_\alpha$, where

$$A_\alpha(u) = \frac{d}{du} \frac{1}{\Gamma(\alpha)} \int_0^u A(v)\,(u - v)^{\alpha-1} dv. \tag{12}$$

In particular, the series

$$\sum_1^\infty A(n)[e^{-\lambda_n s} - e^{-\lambda_{n+1} s}] \tag{13}$$

is absolutely convergent to the sum $f(s)$ in the largest half-plane where either $\mu(\sigma; f) < \frac{1}{2}$ or $\nu(\sigma; f) < 2$.

The first part of the theorem follows from Theorem 2 or its corollary. In order to determine the value of $A_\alpha(u)$ we observe that the series in (7) is summable by typical means $R(\lambda, \kappa)$ for sufficiently large values of κ.[7] It follows that (11) and (12) are valid if α is replaced by $\kappa + 1$. To complete the proof we need merely formula (5) and the property $I^\alpha I^\beta = I^{\alpha+\beta}$ of fractional integrals. A similar theorem holds if the series in (7) is merely asymptotic to $f(s)$ in a suitable domain.

Series (13) is of course obtained from (7) by partial summation, a procedure valid when (7) converges. But we do not know that this series has a half-plane of convergence, nor can we conclude this fact from the absolute convergence of (13). Indeed, it is possible to construct examples of Dirichlet series such that the abscissa of absolute convergence of (13) exceeds the abscissa of convergence of (7) by any given amount. This simple method of summing a Dirichlet series does not seem to have received the attention which it deserves. In the related theory of factorial series partial summation is a recognized method of analytic continuation.

Let us now apply Theorem 3 to Dirichlet series. Ostrowski[8] has proved that the reciprocal of a formal (asymptotic) Dirichlet series is a formal (asymptotic) Dirichlet series. Such results are of course not obtainable by our methods. Nevertheless, our results throw some side light on those of Ostrowski in simple cases. Suppose that

$$f(s) \sim \sum_1^\infty a_n e^{-\lambda_n s}, \; a_1 = 1, \; \lambda_1 = 0, \tag{14}$$

and that $f(s) \subset R_\alpha(a)$. Suppose further that $f(s) \longrightarrow 1$ uniformly in t as $\sigma \longrightarrow \infty$ which is certainly the case if the series has a half-plane of uniform convergence. Suppose finally that $f(s) \neq 0$ for $\sigma \geqq a_1 \geqq a$.

By Theorem 3, $[f(s)]^{-\gamma} \subset R_{1/2+\epsilon}(a_1)$ and, *a fortiori*, $\subset R_1(a_1)$. Since $\mu(\sigma; f^{-\gamma}) = 0$ we can apply Theorem 6 to the formal series

$$[f(s)]^{-\gamma} \sim \sum_1^\infty b_n^{(\gamma)} e^{-\mu_n s}, \tag{15}$$

obtaining

$$[f(s)]^{-\gamma} = \sum_1^\infty B^{(\gamma)}(n) [e^{-\mu_n s} - e^{-\mu_{n+1} s}], \tag{16}$$

the series being absolutely convergent for $\sigma > a_1$.

A similar situation holds in the case of Theorem 4 which is a vague analogue of theorems due to Ritt and Ostrowski.[9] In Ritt's case (also studied by Ostrowski) the coefficients $f_1(s), f_2(s), \ldots, f_m(s)$ are absolutely convergent Dirichlet series, i.e., they belong to every class $R_\epsilon(a)$ for $\epsilon > 0$ and some suitably chosen a. Ritt can then affirm that the roots are representable by absolutely convergent Dirichlet series in some half-plane. All that our theorem asserts is that if a root be regular in a half-plane $\sigma > a_1 \geqq a$, then it belongs to $R_{1/2+\epsilon}(a_1)$. Combining our results with those of Ritt, we conclude that his series can be evaluated by partial summation in the half-plane $\sigma > a_1$.

In the general case of Ostrowski the coefficients are formal Dirichlet series and so are the roots. Our results do not apply to any such case at all. But if we assume in addition that, for $\nu = 1, 2, \ldots, m$, $f_\nu(s) \subset R_{\nu\alpha}(a)$, we can conclude that if a root is regular for $\sigma > a_1 \geqq a$, then it belongs to $R_\mu(a_1)$ for every $\mu > \alpha + \frac{1}{2}$. Moreover, the formal series of Ostrowski for the root is a Dirichlet series in the usual sense, representable by formulas (11) and (12) for $\sigma > a_1$, α being replaced by μ.

[1] These PROCEEDINGS, **19**, 902–908 (1933).

[2] Especially in the summation of such series by the typical means of M. Riesz. See G. H. Hardy and M. Riesz, *The General Theory of Dirichlet's Series*, Cambridge (1915).

[3] S. Bochner, *Vorlesungen über Fouriersche Integrale*, Leipzig (1932), Chapter 7.

[4] E. Landau, "Über den Wertevorrat von $\zeta(s)$ in der Halbebene $\sigma > 1$," *Göttinger Nachrichten*, I: **36**, 81–91, p. 83 (1933). Landau's assumption that $f(z)$ is $O[|y|]$ can be replaced by $O[|y|^k]$, $k > 0$, without affecting either method of proof or conclusion.

[5] See Hardy-Riesz, loc. cit., pp. 14–18.

[6] F. Carlson, "Contributions à la théorie des séries de Dirichlet, Note II," *Arkiv för Mat.*, **19A**, 25 (1926).

[7] See Hardy-Riesz, loc. cit., Theorem 41, p. 53. It should be noted that $2\mu(\sigma; f) \leqq \nu(\sigma; f) \leqq 2\mu(\sigma; f) + 1$.

[8] A. Ostrowski, "Über algebraische Funktionen von Dirichletschen, Reihen," *Math. Z.*, **37**, 98–133, 127–128 (1933).

[9] J. F. Ritt, "Algebraic Combinations of Exponentials," *Trans. Amer. Math. Soc.*, **31**, 654–679 (1929); Ostrowski, loc. cit., especially Theorems D and F.

A LOWER LIMIT FOR THE SPECIES OF A PFAFFIAN SYSTEM

BY JOSEPH MILLER THOMAS

DEPARTMENT OF MATHEMATICS, DUKE UNIVERSITY

Communicated September 5, 1933

The object of this paper is to prove the following

THEOREM. *The species*[1] *of a pfaffian system is at least equal to one-half its rank.*

Let the pfaffian system be

$$\omega^1 = 0, \ \ldots, \ \omega^r = 0.$$

The rank, as defined by Engel,[2] is by virtue of a theorem of Cartan[3] double the largest exponent ρ for which

$$\omega^1 \ldots \omega^r(\lambda_\alpha \omega'^\alpha)^\rho \neq 0, \tag{1}$$

ω' denoting the derived form of ω and α being summed. A system of species σ can be written in the reduced form

$$\omega^\alpha = dx^\alpha + A^\alpha_\beta dx^\beta \ (\alpha = 1, \ldots, r; \ \beta = r+1, \ldots, r+\sigma),$$

whence

$$\omega'^\alpha = dA^\alpha_\beta dx^\beta.$$

Consequently every term of (1) is at least of degree $r + \rho$ in the $r + \sigma$ differentials $dx^1, \ \ldots, \ dx^{r+\sigma}$. If ρ were greater than σ, every term in (1) would vanish and that inequation would be contradicted. Hence, $\sigma \geqq \rho$, and the theorem is proved. The equality always holds in the case of a single equation, and may hold in other cases, as the example below shows.

Another lower limit to the species has been given by Vranceanu,[4] who writes

$$\omega'^\alpha \equiv (\alpha, 1)(\alpha, 2) + \ \ldots \ + (\alpha, 2p_\alpha - 1)(\alpha, 2p_\alpha)$$

mod. $\omega^1, \ \ldots, \ \omega^r$; $(1, 1), \ \ldots, \ (1, 2p_1)$; $\ \ldots$; $(\alpha - 1, 1), \ \ldots, \ (\alpha - 1, 2p_{\alpha-1})$, where the symbols like $(\alpha, 1)$ denote linear forms. If some $\omega' \equiv 0$, the corresponding p is zero. His inequality is

$$\sigma \geqq p_1 + \ \ldots \ + p_r.$$

To compare with that already given, note that the expression

$$\omega^1 \ldots \omega^r(\lambda_\alpha \omega'^\alpha)^{p_1 + \ldots + p_r}$$

contains the non-zero term

$$\omega^1 \ldots \omega^r(\lambda_1 \omega'^1)^{p_1} \ldots (\lambda_r \omega'^r)^{p_r}.$$

Hence

$$\rho \geqq p_1 + \ldots + p_r.$$

The limit given here is therefore always as good as Vranceanu's. That it is sometimes better is seen from the following system whose species is two:

$$\omega^1 = dx^5 + x^1dx^2, \ \omega^2 = dx^6 - x^3dx^1 + x^2dx^4,$$
$$\omega'^1 = dx^1dx^2, \ \omega'^2 = dx^1dx^3 + dx^2dx^4.$$

We have $p_1 = 1$, $p_2 = 0$, $p_1 + p_2 = 1$, whereas $\rho = 2$.

Had the equations been written in the opposite order, we should have found $p_1 = 2$, $p_2 = 0$, $p_1 + p_2 = 2$. This illustrates the fact that $p_1 + \ldots + p_r$, unlike the rank, is not an invariant.

[1] Species is defined in the author's paper "Pfaffian Systems of Species One," *Trans. Amer. Math. Soc.*, **35**, 356–71 (1933).

[2] Cf. Goursat, E., *Problème de Pfaff*, Paris, 291 (1922).

[3] Cartan, E., *Invariants Intégraux*, Paris, 59 (1922).

[4] Vranceanu, G., *Comptes Rendus*, Paris, **196**, 1859–61 (1933).

A PROPERTY OF RECURRING SERIES

By Morgan Ward

Department of Mathematics, California Institute of Technology, Pasadena

Communicated September 6, 1933

1. If

(U): $\qquad U_1, U_2, \ldots, U_n, \ldots$

denotes a sequence of rational numbers satisfying a linear difference equation of order k with rational coefficients, then if a group of k consecutive terms of (U) ever repeats itself, all the roots of the polynomial associated with the difference equation are roots of unity, and the sequence (U) is periodic.[1] I show here that the like occurs, generally speaking, if *one* term of the sequence repeats itself at regular intervals a sufficient number of times. More precisely, I shall show that

If in any particular rational solution (U) of the difference equation

$$\Omega_{n+k} = P_1\Omega_{n+k-1} + P_2\Omega_{n+k-1} + \ldots + P_k\Omega_n, \ P_i \text{ rational}, \ i = 1, \ldots, k, \quad (1)$$

we have

$$U_a = U_{a+b} = U_{a+2b} = \ldots = U_{a+kb} \neq 0, \quad (2)$$

where a, b are fixed positive integers, and if the associated polynomial

$$x^k - P_1x^{k-1} - P_2x^{k-2} - \ldots - P_k \qquad (3)$$

is irreducible in the field of rationals, then the polynomial is cyclotomic, and every solution of the difference equation is periodic.

2. The necessity for the restrictive hypotheses of the theorem is shown by the following two examples.

For the difference equation $\Omega_{n+3} = \Omega_{n+2} + 4\Omega_{n+1} - 4\Omega_n$ the associated polynomial $x^3 - x^2 - 4x + 4$ factors into $(x - 1)(x - 2)(x + 2)$. Therefore, if c is any rational number $\neq 0$, the particular solution $U_n = 2^n + (-2)^n + c\,1^n$ has all terms with odd subscripts equal to c.

On the other hand, for the difference equation $\Omega_{n+4} = \Omega_{n+2} + \Omega_n$, the associated polynomial $x^4 - x^2 - 1$ is irreducible and not cyclotomic, while any particular solution (U) with $U_2 = U_4 = 0$ has all terms with even subscripts equal to zero.

3. The theorem itself may be proved as follows. Let $\alpha_1, \alpha_2, \ldots, \alpha_k$ be the roots of the polynomial (2). Then U_n is of the form

$$U_n = A_1\alpha_1^n + A_2\alpha_2^n + \ldots + A_k\alpha_k^n$$

where the A_i are fixed non-vanishing algebraic numbers. If the common rational value of $U_a, U_{a+b}, \ldots, U_{a+kb}$ in (2) is denoted by c, we have then $k + 1$ homogeneous linear equations in $A_1\alpha_1^a, \ldots, A_k\alpha_k^a$ and c:

$$(A_1\alpha_1^a)\alpha_1^{rb} + (A_2\alpha_2^a)\alpha_2^{rb} + \ldots + (A_k\alpha_k^a)\alpha_k^{rb} - c = 0, r = 0, 1, \ldots, k. \quad (4)$$

Since $c \neq 0$, the determinant of this system must vanish. But this determinant is of Vandermonde's type; we obtain, therefore

$$V(\alpha_1^b, \alpha_2^b, \ldots, \alpha_k^b, 1) = \prod_{i \neq j} (\alpha_i^b - \alpha_j^b) \prod (\alpha_i^b - 1) = 0.$$

Hence for some i, j, we must have either $\alpha_i^b = 1$ or $\alpha_i^b = \alpha_j^b$.

In the first case, $\alpha_1^b = \alpha_2^b = \ldots = \alpha_k^b = 1$ for (3) was assumed irreducible. Therefore every root of (3) is a root of unity, and the theorem follows.

In the second case, on appealing again to the irreducibility of (3), we see that the b^{th} powers of the roots of (3) can be grouped into m sets of l equal powers each, say

$$\alpha_{rl+1}^b = \alpha_{rl+2}^b = \ldots = \alpha_{rl+l}^b = \zeta_{r+1}, r = 0, 1, \ldots, m - 1, ml = k,$$

where $\zeta_1, \zeta_2, \ldots, \zeta_m$ are distinct algebraic numbers.

If $m = 1$, ζ_1 is rational, and on comparing U_a and U_{a+b}, we see that ζ_1 is unity, giving the first case again. The assumption that $m > 1$ leads to a contradiction. For then we obtain from (4) the equations

$$(A_1\alpha_1^a + \ldots + A_l\alpha_l^a)\zeta_1^r + (A_{l+1}\alpha_{l+1}^a + \ldots + A_{2l}\alpha_{2l}^a)\zeta_2^r + \ldots - c = 0,$$
$$r = 0, 1, \ldots, m.$$

Since $c \neq 0$, we infer as before that

$$V(\zeta_1, \zeta_2, \ldots, \zeta_m, 1) = \prod_{i \neq j} (\zeta_i - \zeta_j) \prod (\zeta_i - 1) = 0.$$

This last relation is impossible, for the ζ are all distinct and all irrational.

[1] See E. T. Bell, these PROCEEDINGS, **16**, 750–752 (1930). Such a repetition will certainly occur if the difference equation has rational integral coefficients and the sequence (U) is bounded provided that $U_1, U_2, \ldots, U_k$ are integers, a result which is due to Laguerre.

INTENSITIES IN ATOMIC SPECTRA

BY M. H. JOHNSON, JR.*

DEPARTMENT OF PHYSICS OF NEW YORK UNIVERSITY

Communicated August 8, 1933

Introductory.—In the theory of complex spectra a number of methods have been developed for calculating the relative energies of the states arising from an electronic configuration. In these methods various schemes of coupling together the orbital and spin angular momenta play an important rôle. Thus a common procedure is to find the matrix of the energy in *LS* coupling and to obtain the eigenvalues as the roots of the corresponding secular determinant. The advantage of this and other schemes is principally due to the fact that the total angular momentum **J**, which is an integral of the motion, is given specified values. The energy matrix then takes on an especially simple form (factored according to *J* values). There is a further advantage in that the energy as well as other dynamical quantities can often be calculated rather simply in definite coupling schemes. The particular choice of coupling may be dictated by convenience for the calculation and occasionally by physical consideration, for the actual states of an atom sometimes very nearly correspond to definite coupling arrangement among the angular momentum vectors.

In this paper we are concerned with the matrix of the electric moment whose components determine the intensity of the lines radiated by the atom. The problem naturally divides into two parts: I, the determination of the electric moment in a definite coupling scheme; II, the electric moment matrix in intermediate coupling. The procedure for determining the latter from the former is quite straightforward and will be illustrated by an example.

Intensities in a Definite Coupling Scheme.—Let us suppose the atomic states are obtained by coupling the vectors $\mathbf{l}_1 \ldots \mathbf{l}_N$ $\mathbf{s}_1 \ldots \mathbf{s}_N$ together

in a certain way to give a resultant L and a resultant S which in turn are combined to give $\mathbf{J}$, the total angular momentum of the atom. It will be clear that the method of calculating the electric moment is not limited to LS coupling (which we have chosen for simplicity) but can be used when $\mathbf{l}_1 \ldots \mathbf{s}_N$ are coupled together in any order. Thus we might have

$$\begin{array}{ll} \mathbf{l}_1 + \mathbf{l}_2 = \mathbf{L}_2 & \mathbf{s}_1 + \mathbf{s}_2 = \mathbf{S}_2 \\ \mathbf{L}_2 + \mathbf{l}_3 = \mathbf{L}_3 & \mathbf{S}_2 + \mathbf{s}_3 = \mathbf{S}_3 \\ \cdots\cdots\cdots & \cdots\cdots\cdots \\ \mathbf{L}_{N-1} + \mathbf{l}_N = \mathbf{L} & \mathbf{S}_{N-1} + \mathbf{s}_N = \mathbf{S} \end{array} \qquad \mathbf{L} + \mathbf{S} = \mathbf{J} \qquad (1)$$

The process of coupling may be described as a unitary transformation from a representation in which $l_1^2 \ldots s_N^2 \; l_{1z} \ldots s_{Nz}$ are diagonal to a representation in which $l_1^2 \ldots s_N^2 \; L_2^2 \ldots L^2 \; S_2^2 \ldots S^2 J^2 J_z$ are diagonal. The transformation is factorable into products, each product corresponding to one of Eqs. (1) and representing the coupling of a pair of vectors.

The electric moment is

$$\mathbf{P} = \Sigma_k \mathbf{p}_k \qquad (2)$$

where $\mathbf{p}_k$ is the electric moment of the kth electron which satisfies the familiar commutation relations

$$\begin{aligned} [p_{kx}, \; l_{kx}] &= 0 \\ p[_{ky}, \; l_{kx}] = [l_{ky}, \; p_{kx}] &= -ip_{kz} \end{aligned} \qquad (3)$$

and commutes with all components of $\mathbf{l}_i$, $\mathbf{s}_j (i \neq k)$. It is clear that $\mathbf{p}_k$ satisfies the commutation relations (3) with $\mathbf{L}_i = \Sigma_r \mathbf{l}_r$ if the sum contains $\mathbf{l}_k$. Otherwise it commutes with $\mathbf{L}_i$. Now it has been shown[1] that if $\mathbf{l}_1$ and $\mathbf{l}_2$ are coupled together and $\mathbf{p}_1$ satisfies these commutation relations (3) with $\mathbf{l}_1$, then the transformation which couples $\mathbf{l}_1$ and $\mathbf{l}_2$ together introduces certain factors in the matrix elements of $\mathbf{p}_1$ which we designate by $F(l_1 l_2 L_2)$. (In the above references these factors were designated by $P^{l_1 \; l_2 \; L'_2}_{l'_1 \; l_2 \; L_2} / P^{l_1 \; l_2}_{l'_1 \; l_2}$.) It is clear that as the vectors $\mathbf{l}_1 \ldots \mathbf{l}_{i-1}$ are coupled together, the matrix components of $\mathbf{p}_i$ will simply involve δ functions of the quantum numbers so introduced because p_i commutes with all these vectors. Similarly the coupling of all the $\mathbf{s}$ vectors introduces δ functions of the spin quantum numbers. But when $\mathbf{l}_i$ is coupled to $\mathbf{L}_{i-1}$, the factors $F(l_i L_{i-1} L_i)$ are introduced and at each successive coupling step the factors $F(l_j L_{j-1} L_j)$ $(j > i)$ are introduced. Thus we see when all the coupling processes are completed, the matrix elements of p_{iz} are given by

$$\begin{aligned} &\delta\begin{pmatrix}\text{all spin} \\ \text{qu. numbers}\end{pmatrix} \delta(l_1 l'_1) \ldots \delta(l_{i-1} l'_{i-1}) P^{l_i}_{l'_i} \, \delta(l_{i+1} l'_{i+1}) \ldots \delta(l_N l'_N) \delta(L_2 L'_2) \\ &\quad \ldots \delta(L_{i-1} L'_{i-1}) F(l_i L_{i-1} L_i) \ldots F(l_N L_{N-1} L) F(LSJ) G(JM_J) \end{aligned} \qquad (4)$$

where $P^{l_i}_{l'_i}$ is a radial integral involving the specific nature of the atom

under consideration and $G(JM_J)$ gives the relative probability of the Zeeman components. If we sum the intensities of the Zeeman components, this last factor is replaced by $G(J)$

$$G^2(J) = \begin{cases} {}^1/_3(J+1)(2J+1)(2J+3) & J \longrightarrow J+1 \\ {}^1/_3J(J+1)(2J+1) & J \longrightarrow J \\ {}^1/_3J(2J-1)(2J+1) & J \longrightarrow J-1 \end{cases} \tag{5}$$

We illustrate (4) by obtaining the familiar Kronig[2] intensity formulas. Let $\mathbf{L}_0$ and $\mathbf{S}_0$ be the orbital and spin angular momentum of the ion and $\mathbf{l}$, $\mathbf{s}$ the orbital and spin angular momentum of the added electron. The coupling scheme is

$$\begin{aligned} \mathbf{l} + \mathbf{L}_0 &= \mathbf{L} \\ \mathbf{S} + \mathbf{S}_0 &= \mathbf{S} \\ \mathbf{L} + \mathbf{S} &= \mathbf{J}. \end{aligned} \tag{6}$$

So we see the electric moment matrix components of the jumping electron are $P^l_{l'}F(lL_0L)F(LSJ)G(J)$ and are diagonal in all the quantum numbers, other than L_0, describing the parent term of the ion. The square of this quantity is the intensity of the transition and yields the expected Kronig formulas.

The coupling scheme might have been

$$\begin{aligned} \mathbf{l} + \mathbf{s} &= \mathbf{j} \\ \mathbf{l}_0 + \mathbf{s}_0 &= \mathbf{j}_0 \\ \mathbf{J}_0 + \mathbf{J} &= \mathbf{J}. \end{aligned} \tag{7}$$

In this case the desired matrix components are given by $P^l_{l'}F(l\ j)F(jJ_0J)$ $G(J)$ and are diagonal in the quantum numbers other than J_0 describing the parent term of the ion. If we are dealing with a two electron configuration, $\mathbf{L}_0$ and $\mathbf{S}_0$ refer to the other electron. Our formulas become the same as those used by Bartlett[3] to calculate intensities for two electrons in jj coupling.

We must consider the influence of the Pauli principle. The discussion of this question in a previous paper[4] is not directly applicable here for it is necessary to use distributions of electrons among the states of two electronic configurations whereas in the discussion referred to the distributions were explicitly limited to the states of one electronic configuration. Let the configuration of the initial state be designated by A and the corresponding number of electrons in each subgroup (shell) be $M_1M_2\ldots$. Suppose in the final configuration B an electron has jumped from the subgroup i to the subgroup j. The operator which can cause such transitions is $\Sigma_p(i)\Sigma_r(j)a^+_pa_r$ so that the part of the electric moment operator of interest for this discussion is

$$\mathbf{p}_s = \Sigma_{pr}(i \text{ and } j)a^+_pa_r\mathbf{p}_{pr}. \tag{8}$$

Let
$$M_s = M_i + M_j$$
$$\mathbf{l}_s = \mathbf{l}_i + \mathbf{l}_j. \tag{9}$$

It is clear that M_s is constant for all our distributions and that $\mathbf{p}_s$ satisfies all the usual commutation relations with $\mathbf{l}_s$ and the other $\mathbf{l}$ and $\mathbf{s}$ vectors. This means that our formulas, which are based on these commutation relations, determine the dependence of the electric moment on all coupling quantum numbers which do not refer to i and j, and on the quantum numbers referring to the resultant angular and spin momentum for the two shells i and j.

If $M_s = 1$, that is the jumping electron does not belong to a shell of equivalent electrons in either initial or the final state, then l_s is just the angular momentum of this electron. For if the distribution belongs to A, $\mathbf{l}_j \equiv 0$ whereas if it belongs to B, $\mathbf{l}_i \equiv 0$.

When the jumping electron belongs to a shell of equivalent electrons, the situation is more complicated. It has been shown[5] that if there are two equivalent electrons, the only effect besides the change in the enumeration of the states of LS coupling, is the introduction of a $\sqrt{2}$ in all matrix components. It appears that it is simplest to treat more difficult cases individually by finding the wave functions for the coupled[6] states and calculating the electric moment matrix therefrom.

Intensities in Intermediate Coupling.—Let us now suppose that the energy matrix for the configurations A and B is known in some definite coupling scheme. Let S_A be the transformation that carries the part of the matrix referring to the states of the configuration A to a diagonal form and let S_B carry the part referring to the states of B to a diagonal form. Then the part of the electric moment connecting A and B is given in intermediate coupling, by
$$\mathbf{P}' = \mathbf{S}_A \mathbf{P} \mathbf{S}_B^{-1}. \tag{10}$$

Inasmuch as the energy is factored according to J values, both S_A and S_B are diagonal in J. This fact, together with the theorem of spectroscopic stability, leads at once to the J group sum rule.[7] Also, because of the degeneracy in the energy with respect to M_J, we may sum over the Zeeman components before transforming the electric moment into intermediate coupling. This is accomplished by using Eq. (4) for the matrix elements of the electric moment with $G(JM_J)$ replaced by $G(J)$.

As an example we take the transitions of $p^2 - sp$. The energy matrix in LS coupling is

$J = 2$

	$^1D_2$	$^3D_2$
$^1D_2$	$^1/_{25}F^2$	$(^1/_2)^{1/2}a$
$^3P_2$	$(^1/_2)^{1/2}a$	$-^5/_{25}F^2 + a/2$

$J = 2$

	$^3P_2$
$^3P_2$	$-^1/_3G^1 + a'/2$

$J = 1$

	$^3P_1$
$^3P_1$	$-a/2 - {}^5/_{25}F^2$

$J = 1$

	$^3P_1$	$^1P_1$
$^3P_1$	$-{}^1/_3G^1 - a'/2$	$-({}^1/_2)^{1/2}a'$
$^1P_1$	$-({}^1/_2)^{1/2}a'$	${}^1/_3G^1$

$J = 0$

	$^3P_0$	$^1S_0$
$^3P_0$	$-{}^5/_{25}F^2 - a$	$-(2)^{1/2}$
$^1S_0$	$-(2)^{1/2}a$	${}^{10}/_{25}F^2$

$J = 0$

	$^3P_0$
$^3P_0$	$-{}^1/_3G^1 - a'$

Here the a's are the spin-orbit parameters[8] and the F's and G's are Slater's electrostatic integrals. The electric moment summed over the Zeeman components is from Eq. (4)

P^2 (sp)

	$^3P_2$	$^3P_1$	$^1P_1$	$^3P_0$
$^1D_2$	0	0	$\sqrt{{}^{10}/_3}$	0
$^3P_2$	$\sqrt{{}^5/_2}$	$-\sqrt{{}^5/_6}$	0	0
$^3P_1$	$-\sqrt{{}^5/_6}$	$\sqrt{{}^1/_2}$	0	$-\sqrt{{}^2/_3}$
$^3P_0$	0	$-\sqrt{{}^2/_3}$	0	0
$^1S_0$	0	0	$-\sqrt{{}^2/_3}$	0

We use the $^3P_1$ as a reference level for p^2 and the $^3P_0$ as a reference level for sp. If we let

$$\alpha = {}^1/_{25}F^2/a$$
$$\beta = {}^1/_3G^1/a'$$
$$\epsilon = W/a,\ W/a'$$

the transformation coefficients can all be expressed in terms of the roots of the following three secular equations

$$J = 2 \text{ of } p^2 \qquad (\epsilon_k^2)^2 - ({}^3/_2 + 6\alpha)\epsilon_k^2 - 2 = 0 \qquad k = 1, 2$$
$$J = 1 \text{ of } sp \qquad (\epsilon_k^1)^2 - ({}^3/_2 + 2\beta)\epsilon_k^1 + \beta = 0$$
$$J = 0 \text{ of } p^2 \qquad (\epsilon_k^0)^2 - 15\alpha\epsilon_k^0 - {}^{15}/_2\alpha - {}^9/_4 = 0$$

We find for the electric moment matrix in intermediate coupling

P^2 (sp)

		$^3P_2$	ϵ_k^1	$^3P_0$
	ϵ_k^2	$-\sqrt{\dfrac{5}{2[1 + 2(1 - \epsilon_k)^2}}$	$\dfrac{\sqrt{{}^5/_6} + \sqrt{{}^{40}/_3}(1 - \epsilon_k^2)({}^1/_2 - \epsilon_k^1)}{\sqrt{[1 + 2(1 - \epsilon_k^2)^2][1 + 2({}^1/_2 - \epsilon_k^1)^2]}}$	0
$^3P_1$		$-\sqrt{{}^5/_6}$	$\dfrac{\sqrt{{}^1/_2}}{\sqrt{1 + 2({}^1/_2 - \epsilon_k^1)^2}}$	$-\sqrt{{}^2/_3}$
	ϵ_k^0	0	$\dfrac{-\sqrt{{}^2/_3}[-1 + ({}^1/_2 + \epsilon_k^0)({}^1/_2 - \epsilon_k^1)]}{\sqrt{[1 + 2({}^1/_2 - \epsilon_k^1)^2][1 + {}^1/_2({}^1/_2 + \epsilon_k^0)^2]}}$	0

The states affected by the coupling are here designated by their energy. The square of these matrix elements gives the relative intensities of the lines arising from the states of intermediate coupling.

The writer wishes to thank Professor Breit for encouragement on the subject of this paper. He is also indebted to Harvard University for a Whiting Fellowship which he held when this work was begun.

Transition*	$F(RST)$
$R \longrightarrow R+1$, $T \longrightarrow T+1$	$\left\{\frac{(R+S+T+2)(T+R-S+1)(T+R-S+2)(R+S+T+3)}{4(T+1)^2(2T+1)(2T+3)}\right\}^{1/2}$
$R \longrightarrow R+1$, $T \longrightarrow T$	$\left\{\frac{(R+S-T+1)(T-R+S)(T+R-S+1)(R+S+T+2)}{4T^2(T+1)^2}\right\}^{1/2}$
$R \longrightarrow R+1$, $T \longrightarrow T-1$	$-\left\{\frac{(R+S-T+1)(R+S-T+2)(T-R+S)(T-R+S-1)}{4T^2(2T-1)(2T+1)}\right\}^{1/2}$
$R \longrightarrow R$, $T \longrightarrow T+1$	$-\left\{\frac{(R+S+T+2)(R+S-T)(T+1+R-S)(T+1+S-R)}{4(T+1)^2(2T+1)(2T+3)}\right\}^{1/2}$
$R \longrightarrow R$, $T \longrightarrow T$	$\frac{T(T+1)-S(S+1)+R(R+1)}{2T(T+1)}$
$R \longrightarrow R$, $T \longrightarrow T-1$	$-\left\{\frac{(R+S+T+1)(R+S+1-T)(T+R-S)(T+S-R)}{4T^2(2T-1)(2T+1)}\right\}^{1/2}$
$R \longrightarrow R-1$, $T \longrightarrow T+1$	$-\left\{\frac{(R+S-T-1)(R+S-T)(T-R+S+2)(T-R+S+1)}{4(T+1)^2(2T+1)(2T+3)}\right\}^{1/2}$
$R \longrightarrow R-1$, $T \longrightarrow T$	$\left\{\frac{(R+S-T)(T-R+S+1)(T+R-S)(R+S+T+1)}{4T^2(T+1)^2}\right\}^{1/2}$
$R \longrightarrow R-1$, $T \longrightarrow T-1$	$\left\{\frac{(R+S+T)(T+R-S-1)(T+R-S)(R+S+T+1)}{4T^2(2T-1)(2T+1)}\right\}^{1/2}$

* The selection rules are $\Delta T = \pm 1, 0$, $\Delta R = \pm 1, 0$, $\Delta S = 0$ and the transitions $0 \longrightarrow 0$ are forbidden.

* NATIONAL RESEARCH FELLOW.

[1] Guttinger and Pauli, *Zeits. Phys.*, **67**, 743 (1931); M. H. Johnson, Jr., *Phys. Rev.*, **38**, 1628 (1931); see Appendix for a table of these factors.

[2] Kronig, *Zeits. Phys.*, **31**, 885 (1925); **33**, 261 (1925).

[3] J. H. Bartlett, *Phys. Rev.*, **34**, 1247 (1929).

[4] M. H. Johnson, Jr., *Ibid.*, **43**, 627 (1933).

[5] G. H. Shortley, *Ibid.*, **40**, 185 (1932).

[6] C. W. Ufford, *Ibid.*, **40**, 974 (1933).

[7] Harrison and Johnson, *Ibid.*, **38**, 757 (1931).

[8] The a's are essentially the average value of $1/r\ dV/dr$.

THE BIOGEOGRAPHIC RELATIONS OF THE ORBITOID FORAMINIFERA

By Thomas Wayland Vaughan

Scripps Institution of Oceanography, University of California, La Jolla, California

Communicated August 20, 1933

Introduction.—During the past few years I have published many papers on the orbitoid foraminifera of America, and research assistants at the Scripps Institution are the authors of other papers, including two by G. L. Whipple, one on the Eocene larger foraminifera of Eua, Tonga Islands, and the other on the Miocene larger foraminifera of Viti Levu, Fiji Islands. Of the latter paper only an abstract has been published. One large paper by me on the genus *Lepidocyclina*, and one by D. W. Gravell on the Tertiary larger foraminfera of Venezuela are in press in the Smithsonian Miscellaneous Collections. Another paper by me on the Tertiary larger foraminifera of Ecuador is awaiting publication as a chapter in a volume by Dr. George Sheppard on the geology of Ecuador. I have just completed a revised statement regarding the orbitoids for inclusion in a second edition of Dr. Joseph A. Cushman's "Foraminifera, their classification and economic use." As a result of the studies above indicated consideration has again been given to both the stratigraphic and geographic distribution of the orbitoids, that is, their paleogeographic distribution and its probable significance. This study is also an extension of my former studies of the zoogeography of the coral faunas of the Indo-Pacific region (Vaughan, 1907, 1917, 1918, 1919) and of three symposia on the correlation of Tertiary geological formations organized by me, viz., First Pan-Pacific Science Conference, Honolulu, 1920; Second Pan-Pacific Science Congress, Melbourne and Sydney, 1923; Paleontological Society, Washington, D. C., 1923.

Although the results obtained from the present reconsideration of the biogeographic significance of the orbitoids are not conclusive, a comprehensive statement of the factors that enter into the problem, the presentation of a possible or probable solution of it, and the indication of needed additional specific researches seem desirable.

Essential Features of the Orbitoids.—The orbitoid foraminifera have discoid, lenticular, selliform, stellate or triangular tests. Their range in diameter is from 1 mm. to nearly 100 mm., but between 5 and 20 mm. in diameter is the more usual size. They all have a median zone of chambers, called the equatorial chambers. Except in one genus, *Omphalocyclus*, there are lateral chambers over the center of the test and these decrease in number toward the periphery. In *Omphalocyclus* there are no lateral

chambers over the center of the test, and the number of layers of chambers increases toward the periphery, thereby producing a biconcave disc. The initial chambers of specimens belonging to the microspheric generation form a spiral, those of megalospheric specimens form a spiral in some, but not all, genera. No characterization of the different genera will be attempted here. Most of the generic definitions are contained in the first edition of Cushman's "Foraminifera, their classification and economic use." A few more are given in Galloway's "Manual of Foraminifera," and the latest treatment is the one by me in the revised edition of Cushman's work above mentioned. The list in the present paper differs from the text in Cushman's work in that *Miolepidocyclina* is here treated as a valid genus, while in my paper on the Orbitoididae it is placed in the synonymy of the genus *Miogypsina*. Although I think that *Miolepidocyclina* should be referred to the synonymy of *Miogypsina*, as both European and American species can be assigned to it, its use is convenient for biogeographical purposes.

A name proposed by Silvestri, *Astrolepidina* (Silvestri, 1931), as a subgenus of *Lepidocyclina*, is not entered in my lists, because he named no type-species of the subgenus and distinguished it from the other subgenera of *Lepidocyclina* solely by its stellate form. Since there are some stellate species of *Lepidocyclina* which have in the megalospheric generation embryonic chambers similar to those of the subgenus *Nephrolepidina* and other species have embryonic chambers similar to those of *Lepidocyclina* the definition is inadequate. The desirability of erecting subgeneric names for these stellate forms is questionable. Sectional designations appear preferable.

Monolepidorbis (Astre, 1927), is here listed under the orbitoids, although it is probably more closely related to *Linderina* than to them. However, it is an affiliate of the orbitoids. It is omitted from my paper on the orbitoids in Cushman's revised edition of his "Foraminifera."

Ecologic Relations of the Orbitoids.—The orbitoids occur in temperate, subtropical and tropical latitudes, often in association with corals of reef facies and with the nullipore (lithothamnium) type of calcareous algae. In many places, as in the southern parts of Georgia and Alabama and over much of Florida, they are so abundant as to be the principal constituents of some of the limestones.

There are now living in the sea foraminifera that are similar in form and size to the orbitoids, although none are so large as the largest orbitoids. Among such genera are *Cycloclypeus* (Miocene to Recent), *Calcarina* (Cretaceous to Recent), *Siderolites* (Cretaceous to Recent), *Baculogypsina* (Tertiary and Recent), *Archaias* (Miocene to Recent), *Sorites* (Miocene to Recent), *Amphisorus* (Miocene to Recent), and *Marginopora* (Recent). These genera represent several different families, but they are relatively

large and they are of discoid or stellate form. Some of them lived in association with orbitoids.

Other genera of larger foraminifera which are still living and which also lived during Tertiary time and were associated with orbitoids, are the free-living forms, *Operculina* (Cretaceous to Recent) and *Heterostegina* (Eocene to Recent); and the incrusting form, *Carpenteria* (Cretaceous to Recent).

All of the living genera of foraminifera which either simulate the growth-forms of the orbitoids or which were of large size and lived in association with them are characteristic constituents of present-day shoal-water foraminiferal faunas of the tropics and subtropics.

Therefore, because of their association with corals of reef-facies and nullipores, their similarity in size and form to living foraminfera which are characteristic of the shoal-waters of the tropics and subtropics, and the fact that the living representatives of other genera of large size which were associated with the orbitoids are also confined to the shoal-waters of the tropics and subtropics, the evidence seems conclusive that the environment of the orbitoids was warm, shallow water.

TABLE 1

STRATIGRAPHIC AND GEOGRAPHIC DISTRIBUTION OF THE ORBITOIDS

	CRET.	EOC.	OLIG.	MIOC.	EUR.	INDIA	E. IND. & PAC. IDS.	AMER.
Lepidorbitoides Silvestri	x				x	x		
Clypeorbis H. Douvillé	x				x			
Orbitoides d'Orbigny	x				x	x		x
Simplorbites de Gregorio	x				x			
Pseudorbitoides H. Douvillé	x							x
Orbitocyclina Vaughan	x							x
Asterorbis Vaughan and Cole	x							x
Omphalocyclus Bronn	x				x	x		x
Actinosiphon Vaughan		x						x
Lepidocyclina Gümbel								
Polylepidina Vaughan		x						x
Pliolepidina H. Douvillé		x						x
New subgenus, Whipple				x			x	
Multilepidina Hanzawa				x			x	
Lepidocyclina Gümbel		x	x	x			x	x
Nephrolepidina H. Douvillé		x	x	x	x	x	x	x
Eulepidina H. Douvillé			x	x	x	x	x	x
Helicolepidina Tobler		x						x
Miogypsina Sacco			x	x	x		x	x
Miolepidocyclina Silvestri			x	x	x			x
Miogypsinoides Yabe and Hanzawa				x			x	
Discocyclina Gümbel								
Discocyclina Gümbel		x		x	x	x	x	x
Actinocyclina Gümbel		x			x			x
Asterocyclina Gümbel		x			x		x	x
Monolepidorbis Astre	x				x			

Stratigraphic and Geographic Distribution of the Orbitoids.—Table 1 presents in summary form the more important data on this subject.

Notes should be made on two points regarding the stratigraphic distribution as shown in the accompanying tables. The first is that I have not followed Professor Checchia-Rispoli (1925) in his opinion that *Orbitoides*, *Lepidocyclina* and *Miogypsina* occur in association with *Discocyclina* ("*Orthophragmina*") and *Chapmania* in beds of Eocene age in the Appenino Pugliese. The mixture is one bed of organisms elsewhere characteristic of Upper Cretaceous, Eocene and Miocene indicates rather transport and redisposition than that the organisms lived together. The Pliocene of the Waccamaw district in the Carolinas, in the United States, contains excellent specimens of Upper Cretaceous fossils.

The second point concerns the use of the terms Oligocene and Miocene. In Europe it appears that it has been possible to reach consensus of opinion regarding a boundary between the Oligocene and Miocene, but in America, that is, in the Gulf States of the United States, the West Indies, Mexico, Central America and northern South America, there is doubt as to just where the division should be made. Apparently there is no actual stratigraphic break.

There is also doubt regarding the applicability of the European stratigraphic terms in the East Indian region, Australia and the Pacific Islands. I have considered this perplexity in my 1921 papers. The most recent discussion of it is by van der Vlerk (1928).

Under the circumstances, I have endeavored to follow for Europe the accepted European usage; but for America and the western Pacific no attempt has been made to distinguish between upper Oligocene and lower Miocene. The indicated stratigraphic distribution for the latter regions are in terms of the European time scale only approximate, not precise.

TABLE 2

GEOGRAPHIC DISTRIBUTION OF ORBITOIDS DURING THE UPPER CRETACEOUS

Common to Europe, India, America	Orbitoides Omphalocyclus
Common to Europe and India	Lepidorbitoides
Restricted to Europe	Clypeorbis Simplorbites Monolepidorbis
Restricted to America	Pseudorbitoides Orbitocyclina Asterorbis

Means of Geographic Distribution of Foraminifera.—Since the orbitoids were bottom living organisms, although they probably were capable

of limited locomotion, it appears that wide geographic distribution can be explained only be postulating planktonic larval stages. At this stage in this argument we need to consider the problem of the reproductive cycle in foraminifera. Basing opinions on the work of Lister and Schaudinn, the belief has been current for some time that between the asexual reproduction represented by the megalospheric generation there intervenes a generation of zoospore production and zoospore conjugation. As a result of zoospore conjugation the microspheric generation is produced, which by asexual reproduction gives rise to megalospheric individuals.

TABLE 3

GEOGRAPHIC DISTRIBUTION OF ORBITOIDS DURING THE EOCENE

Common to Europe, India, Pacific Islands, America	Discocyclina (Discocyclina) (Asterocyclina)
Common to Europe and America	Discocyclina (Actinocyclina)
Restricted to Europe	None
Restricted to the East Indies and Pacific Islands	None
Restricted to America	Actinosiphon Lepidocyclina (Polylepidina) (Pliolepidina) (Lepidocyclina) (Nephrolepidina) Helicolepidina

TABLE 4

GEOGRAPHIC DISTRIBUTION OF ORBITOIDS DURING THE OLIGOCENE AND MIOCENE

Common to Europe, India, Pacific Islands, America	Lepidocyclina (Nephrolepidina) (Eulepidina) Miogypsina
Common to Europe and America	Miolepidocyclina
Common to East Indies and America	Lepidocyclina
Restricted to Europe	None
Restricted to America	None
Restricted to East Indies, Pacific Islands	Lepidocyclina (Multilepidina) Subg. nov., Whipple Miogypsinoides

Lister (1903) says regarding *Elphidium* ("*Polystomella*") (p. 72),

"Though the zoospores have been seen issuing from the shell, their precise characters, when ripe, have not been accurately described; nor have we as yet direct evidence as to their fate."

On page 77, at the end of his discussion of the relation between the microspheric and megalospheric forms, he says:

"While the megalospheric generation arises asexually, either from a microspheric or a megalospheric parent, it is probable that the microspheric generation arises sexually—i.e., by the conjugation of two similar zoospores."

Schaudinn (1903) reported the conjugation of zoospores. Lister (1907) translated into English a summary statement of Schaudinn's results. The last two sentences of this translation (p. 591) read as follows:

"I have cultivated the young microspheric individuals in a moist chamber as far as the five-chambered stage, when they died, probably from want of nourishment. In most cases the nucleus had repeatedly divided. From these small, many-nucleated microspheric individuals the youngest many-chambered stage described in my earlier publications directly proceeds, so that the life-cycle of *Polystomella* is now complete."

It would look as if the problem were solved, but the validity of the interpretation of Lister and Schaudinn has been attacked, and, therefore, additional investigations were needed.

In an attempt to solve this problem, an investigation of the life history of local species of foraminifera has been undertaken as a part of the scientific program of the Scripps Institution by Mr. Earl H. Myers, and some of his results are given in a paper by him in this number of this Journal. Myers has reared the asexual generation of eight different species; plastogamically produced individuals have been reared in at least two species; two of the species investigated have produced "zoospores." In one species, *Tretomphalus bulloides*, the individuals, after they have developed a float-chamber from a *Discorbis*-like earlier stage, come to the surface, and then within one day they discharge "zoospores." The individuals may be solitary and discharge their "zoospores" directly into the water; or two or more individuals may be drawn together by pseudopodia, so that their apertural faces are in juxtaposition, and then discharge their "zoospores," the "zoospores" from two or more parents thereby becoming mingled. In the other species, *Discorbis patelliformis*, "zoospores" are produced only after two or three individuals have fused (Myers, 1933). This species is considered in more detail by Mr. Myers in his paper in this number of this Journal. The details of the cytological phenomena involved have not been completely worked out and the fate of the "zoospores" has not been positively ascertained. Although the evidence is not yet definite and conclusive, still the assumption of the production of pelagic larval young among many foraminifera seems reasonable, and it also seems reasonable to assume that such larvae can be transported for considerable distances by ocean currents.

With reference to the orbitoids it is necessary to make a guess which accords with such facts as we possess. This guess is that reproduction in

the orbitoids was by means of zoospores and that larvae so produced were transported considerable distances by currents. Such an hypothesis as this needs testing. The suggested tests might be made by rearing the sexual generation of some of the genera already mentioned, viz., *Archaias*, *Soriles*, *Amphisorus*, *Marginopora*, *Calcarina*, *Siderolites*, *Baculogypsina*, *Operculinella*, *Operculina*, *Heterostegina* and *Cycloclypeus*.

Results obtained from the study of corals bear on this problem. In my monograph, "Recent Madreporaria of the Hawaiian Islands and Laysan" (Vaughan, 1907, p. 47), I pointed out that the shallow-water Hawaiian fauna is merely a part of the Indo-Pacific fauna. Some species range from the Red Sea to the Hawaiian Islands. In my experiments on the duration of the larval stages in some of the species of the Florida coral-reef tract, it was shown that planulae would remain in the free-swimming stage from 3 to 23 days before becoming fixed and that they would then develop into normal colonies. The results of my 1910 experiments are expressed in the following table (Vaughan, 1911).

TABLE 5

DURATION OF THE FREE-SWIMMING LARVAL STAGE IN EXPERIMENTS OF 1910

NAME OF SPECIES	DURATION OF FREE-SWIMMING LARVAL STAGE
Favia fragum (Esper)	6–23 days
Agaricia purpurea (LeS.)	11–17 days
Porites clavaria Lam.	12–20 days
astreoides Lam.	7 or 8–22 days

It is pertinent to remark that species of corals belonging to each of the three genera listed above are found as fossils in association with orbitoids.

Dr. C. H. Edmondson, in his papers on the ecology and growth rate of Hawaiian corals (Edmondson, 1928, 1929), gives the following data on the duration of the free-swimming larval stages in Hawaiian corals:

TABLE 6

DURATION OF THE FREE-SWIMMING LARVAL STAGE IN HAWAIIAN CORALS

NAME OF SPECIES	DURATION OF FREE-SWIMMING LARVAL STAGE
Pocillopora cespitosa Dana	Up to 30 days
Cyphastrea ocellina Dana	3–18 days
Dendrophyllia manni Verrill	Up to 30 days

It should be mentioned that coral colonies attached to floating objects, such as pieces of wood and pumice, may be transported from one place to another by currents but, without stating the reasons for my opinion, it seems that the principal means of effecting wide distribution is by floating larvae.

The fact of the wide distribution of many species of corals is adequately

explained by what is known regarding the possibility of larvae being carried by currents hundreds of miles, even a thousand miles, before settling to the bottom. It appears that wide distribution by similar means also occurs in the foraminifera.

Systems of Surface Currents in the Atlantic and Pacific Oceans within Tropical and Subtropical Latitudes.—It is a matter of common knowledge that in both the Atlantic and the Pacific there is a vortex on each side of the equatorial regions. On the north side of the Equator there are the westwardly flowing north equatorial currents. The westwardly flowing south equatorial currents reach to or even a little north of the Equator. In the Atlantic the south equatorial current splits on Cape San Roque and sends a branch northward to join the north equatorial current. As the continents on the west sides of the ocean basins are approached the direction of the flow of the water is diverted to the right in the northern and to the left in the southern hemisphere, until the motion becomes eastward in direction in each hemisphere. The diversion to the right in the northern and to the left in the southern hemisphere continues until a gyral is completed in each hemisphere in each ocean. There is in the Pacific a well-developed equatorial counter current between the north and south equatorial currents, but in the Atlantic the equatorial counter current is weakly developed except on the east side, west of the Gulf of Guinea. This system of currents renders possible the transport of planktonic organisms over long distances within tropical and adjacent temperate latitudes.

In places there are opportunities outside this major circulation for transport within circumscribed regions. For example, there is at least during a part of each year, a gyral surface circulation in the South China Sea. Water moves northward along the west side of the Philippines and southward along the east coast of southern China and French Indo-China.

The distances to which organisms may be carried by such currents depends upon the duration of the pelagic stages in their life-histories. As regards bottom-living organisms, if their larvae have pelagic stages of long duration they may attain wide distribution, especially if there are intervening areas of shoal-water which may serve as landing stages. But if their pelagic stages are of short duration, the distribution of the organisms would be much more restricted.

Interoceanic Connections between the Atlantic and Pacific during Cretaceous and Tertiary Time.—That there was connection by way of the ancient Tethys between southern Europe and the sea southeast of Asia during Cretaceous time is generally admitted, and such connection persisted into Eocene time. For the later history reference will be made to papers by K. Martin (1914, 1919, 1921), Guy E. Pilgrim (1926) and J. H. F. Umbgrove (1929). Martin was the first to reach the conclusion that the Indo-

Pacific region has been definitely detached from Tethys since later Eocene time and that subsequent to that time the developments there have been autochthonous. Pilgrim would push the severance back to lower Eocene time, but indicates connection between an eastern remnant of Tethys with the East Indian region by sea around the southern end of India. That there must have been connection during lower Eocene time between Tethys and Europe is shown by the similarity of the fauna of the lower Eocene Laki series of Sind and Baluchistan and the lower Eocene of southern France (Nuttall, 1925). Umbgrove in his paper (1929) says that new data seem to indicate that during Oligocene time marine connection between the East Indian region and Europe was restored by way of British India.

The possibility of migration around the southern end of Africa during Oligocene and Miocene time should be considered. Orbitoids are found in the upper Oligocene or lower Miocene of both Australia and New Zealand in latitudes equivalent to or higher than the latitude of Cape of Good Hope. In the northern hemisphere they extend in Europe as far north as latitude 45°N., ten degrees farther from the Equator than the Cape of Good Hope. The determining factors in such a migration would be temperature and currents. For the needed information reference may be made to Schott's "Geographie dès Atlantischen Ozeans" (Schott, 1912).

TABLE 7

YEARLY AVERAGE OF WATER TEMPERATURE °C.

LAT. N.	70°	60°	50°	40°	30°	20°	10°	0°	10°	20°	30°	40°	50°	LAT. S. 60°
1. American side	−1.5	3.5	4.0	11.5	25.5	27.0	27.5	26.5	26.0	24.5	20.5	15.0	8.0	3.5
2. Europe-Africa side	6.0	9.0	12.5	16.5	18.0	20.0	26.5	27.0	23.5	17.5	16.0	14.0	2.5	−1.0
Difference 2−1	+7.5	+5.5	+8.5	+5.0	−7.5	−7.0	−1.0	+0.5	−2.5	−7.0	−4.5	−1.0	−5.5	−4.5

Plate 10 of Schott's work presents the averages of temperature data for different periods of the year. North of the Equator the temperature probably would not be too low, but south of latitude about 8°S. the temperature near shore during the cold-period of the year drops to 12.4°C. (54.8°F.). This temperature is probably lower than that which tropical and subtropical organisms could survive. Furthermore, if an inference may be based upon comparing Schott's charts with my own study of the temperature of the Florida coral-reef tract (Vaughan, 1918) the temperature at times may drop below 12.4°C. According to Schott the temperature for the east coast of southern Florida during the winter is above 20°C. In the temperature records published by me there are many which are lower than 20°C. during the winter season. The lowest record was 15.6°C.

at Fowey Rocks, which are just north of the limit of present vigorous growth of reef-corals. This limit is about Carysport Reef, where a minimum winter temperature of 18.15°C. has been recorded. Table 7 copied from Schott is significant in this connection.

It would therefore seem that the low temperatures along the west coast of southern Africa would inhibit migration of orbitoids around the southern end of Africa.

Plate 16 in Schott's work, already cited, presents the surface current systems of the Atlantic during the northern winter. The Benguela current sets northward along the southwest African coast and the Agulhas current sets westward along and off the south end of the continent. The currents, therefore, would oppose migration by a south African route.

From the foregoing comments it appears that, because of both low temperature and adverse currents, migration around the southern end of Africa was not practicable. However, it would seem that further consideration of such a possibility is desirable. Explanations must be found for the presence of a species of *Miogypsina* in the Miocene strata of North Island, New Zealand, so similar to *Miogypsina irregularis* (Micht), common in the lower Miocene of northern Italy and southern France, that Chapman identifies the New Zealand species as *M. irregularis*. Perhaps there may have been migration along the Oligocene route suggested by Umbgrove. Perhaps the dating of the different events involved in the attempt to reconstruct the history may not yet be sufficiently precise. The Oligocene route may have existed during lower Miocene time, or the beds classified as lower Miocene may be of Oligocene age. The puzzle must be left in this condition.

As for the intercommunications between the Atlantic and Pacific across Central America during Cretaceous and Tertiary time, reference will be made to one of my own papers (Vaughan, 1921). Apparently there was no connection during either Lower or Upper Cretaceous time across Central America. There was such connection during middle and upper Eocene, probably all of Oligocene, and most of Miocene time. Whether or no there was connection during early Eocene and early Pliocene time is not conclusively established. Since these uncertainties are not of significance for the problem being considered here, they will not be discussed in this connection.

Probable Routes of Migration of the Orbitoids.—Upper cretaceous. The three genera, *Lepidorbitoides*, *Orbitoides* and *Omphalocyclus* could have migrated between Europe and India along the littoral of Tethys and their occurrence in both regions presents no difficulty. Whether *Orbitoides* and *Omphalocyclus* went from America to Europe or vice versa, there must have been shoals between the two continental areas at that time. The absence of *Orbitoides* and associated genera on the Pacific side

of America is in accord with the conclusion drawn from other evidence that during the Upper Cretaceous, the Atlantic and Pacific basins were separated by land between North and South America.

Eocene.—Tethys during a considerable part of Eocene time furnished a route for the migrations of marine organisms from the Mediterranean region to the tropical Pacific. As the Eocene orbitoids of the Tonga Islands might have been derived from the East Indian region it is not necessary to appeal to landing stages between America and that part of the Pacific to account for the presence there of the orbitoid fauna of Eua. The existence of interoceanic connection between the Atlantic and Pacific in the American Isthmian region during the Eocene presents an easy explanation of the migration of *Discocyclina* (*Discocyclina*) and *D.* (*Asterocyclina*) to the Pacific side of America. The presence of all the subgenera of *Discocyclina* in both America and southern Europe requires the postulate of landing stages across the Atlantic. *Lepidocyclina* and its allies appear to have originated in America during middle and upper Eocene time, and subsequent to that time to have migrated to Europe.

Oligocene and Miocene.—By inspection of the table showing the distribution of orbitoids during this time it is seen that every known genus and subgenus, except two subgenera, *Multilepidina*, recently described by Hanzawa, and another described by Whipple from the Miocene of Vitilevu, but not yet published, and *Miogypsinoides* is found in America. Most of the subgenera of *Lepidocyclina* were present in America during the Eocene and one subgenus, *Polylepidina*, occurs in the lower middle Eocene. *Miogypsina*, as represented by *M. bracuensis* Vaughan in Jamaica, may be older in America than in Europe, the East Indies and the Pacific Islands, and *Miolepidocyclina* may be older in America than elsewhere. For *Lepidocyclina*, the evidence favors tropical or subtropical America being the site of its origin, and both *Miogypsina* and *Miolepidocyclina* may have originated in America.

If the opinions regarding the closing of Tethys are correct, the distribution of the orbitoids during Oligocene and Miocene time was almost certainly from America to Europe and it seems probable that it was from America to tropical Pacific Islands and thence to tropical and subtropical eastern Asia, New Zealand and Australia. The possibility of migration from Europe during Oligocene time, as has been suggested by Umbgrove, should be borne in mind. Migrations across the Atlantic and Pacific, as suggested above, meant the presence of shoals across both the Atlantic and Pacific within tropical or subtropical latitudes during Oligocene and probably during Miocene time. As to how close together the shoals must have been only very rough estimates are possible because the duration of the larval stages of the organisms is not known. However, if the results obtained from the study of corals may be applied to foraminifera, the

intervening distances should be of the order of magnitude of a few hundred to a thousand miles.

Here an erroneous identification should be corrected. Marshall, in his paper on Mangaia (Marshall, 1927, p. 26) says,

> "The third specimen (I b 3) of this series, collected 80 feet above sea level, is a sandy limestone which contains a Lepidocycline of thin, discoid form, somewhat resembling *Lepidocyclina proteiformis* Vaughan."

L. proteiformis belongs to the subgenus *Polylepidina* Vaughan, which until now is known only in the middle and upper Eocene of America. Joleaud (1931, a,b), uses Marshall's mention of *L. proteiformis* as evidence of the presence of *Polylepidina* in the Pacific islands. The other species of *Lepidocyclina* recorded from Mangaia represent a much younger age. Doctor Marshall kindly sent me his thin sections for examination. Without more material I cannot identify the species, but I can say that the species is not *L. proteiformis*.

Bearing of the Distribution of the Orbitoids on the Problem of Former Land Areas in the Atlantic and Pacific Basins.—There are two schools of paleogeographers, one of which adheres to the Dana hypothesis of the permanency of the ocean basins and the continental masses and accepts the conclusions of Hayford, Bowie and their followers regarding isostasy; the other believes in former land-bridges across areas now occupied by abyssal oceanic waters. Adherents to the latter school are Prof. J. W. Gregory and Dr. H. von Ihering, the recent deaths of both of whom we lament, and Prof. L. Joleaud. Papers by them are listed in the bibliography at the end of this paper.

For the organisms discussed in this paper, also for corals, great land-bridges are not necessary for them to have achieved their known geographic distribution, but series of shoals across oceanic areas must be postulated. Certain apparent possibilities in both the Atlantic and Pacific will be briefly examined.

With reference to the Atlantic, there are off the west coast of Africa, Madeira, Cape Verde and Canary Islands, and along the Atlantic Swell there are the Azore, St. Paul and Ascencion Islands. The meager living coral faunas of Cape Verde Islands and the Gulf of Guinea are almost identical with the living Antillenan fauna. Off the Brazilian coast there are Trinidade and Fernando Noronha. Farther north there are Banco Nicktheroy, the Antilles and the Bermudas. There is a possibility that there are shoal areas along the Atlantic Swell which have either not yet been discovered or have not become generally known. If parts of the Atlantic Swell stood within about 100 meters of the surface, it is probable that the requirements for the migration of the orbitoids would be satisfied. Some of the corals would need shallower water, slightly less than 50 meters.

But to account for the distribution of the organisms here considered it is not necessary to postulate extensive mid-Atlantic land areas.

Within the tropical and subtropical areas of the Pacific there are series of islands and associated shoals as far eastward as Easter and Sala y Gomez Islands and the shoals above which they rise, lat. 26–27°30′S., long. 105–110°W. There are in a more southerly latitude, between Easter Island and the South American coast, Podesta, Juan Fernandez and other islands. East of Sala y Gomez, in about the same latitude, St. Felix and St. Ambrose Islands rise from another bank. According to Schott and Schu (1910) the average surface temperature at Juan Fernandez is between 15° and 16°C. The minimum temperatures would probably be too low for orbitoids. At St. Felix and St. Ambrose the average is 18°C., which is probably near the lower limit of temperature endurance of orbitoids, and the minimum might be too low. But these islands are not excluded as a possible landing stage for migration from the west toward the east. Between Easter Island and the Galapagos, lat. 0°, long. 89–93°W., the temperature conditions, between 21° and 23°C., are favorable, but the distance from Sala y Gomez is rather great. The shoal-water coral fauna of Easter Island is of Indo-Pacific affinities. I have not seen any published account of the littoral coral fauna of the Galapago Islands, but I state in my memoir on Hawaiian corals (Vaughan, 1907, p. 47), that "The Panamic fauna is very close to, or the same as, the Galapagos fauna, and is as closely related to the South Pacific fauna as to the Hawaiian." This statement was probably based on a small collection from the Galapagos Islands in the United States National Museum. The point of these remarks is that the living Indo-Pacific shoal-water coral fauna is represented at Easter Island, the Galapagos Islands, and on the Pacific side of Central America.

North of the Equator, between the Hawaiian Islands and the North American mainland, there are Guadalupe Island and Alijos Rocks, and farther south, off the coast of Lower California, Clarion Island and the Revillagigedo group. Clipperton Island is still farther south, lat. 10°N. The distance from Clarion Island to the Island of Hawaii is about 2200 miles, almost the same distance as that from the Island of Oahu to San Francisco. Southwest of Clipperton Island there is a row of three reported banks. Notwithstanding the long distance from the Central American and Mexican coasts to the Hawaiian Islands and the small number of known possible landing places for species of corals, the shoal-water coral faunas of the two regions both belong to the Indo-Pacific fauna, and one species, *Psammocora stellata* (Verrill) is common to Panama, the Hawaiian Islands and the Fiji Islands.

Even under present conditions the same type of coral faunas has spread over the eastern, central and western Pacific. Although most, perhaps

all, of the coral fauna on the Pacific side of Central America might have been derived from migrants from the Caribbean region during Oligocene and Miocene time (Vaughan, 1917) it would be necessary to explain how the species of *Psammocora*, above mentioned, was distributed in Panama and the Hawaiian and Fiji Islands.

It is possible that there are still undiscovered shoals in the Pacific. Every one who is accustomed to using hydrographic charts knows the abbreviation, "E. D.," existence doubtful. The shoal in lat. 7°S., long. about 99°48'W., southwest of the Galapagos Islands, between them and Sala y Gomez, is an instance of a probable shoal along a possible migration route.

Recent exploration of the bottom configuration of the Central Pacific by the Carnegie, by Professor Vening Meinesz in his cruise across the Pacific in a Netherlands submarine, and by the U. S. S. Ramapo, in command of Capt. Claude Mayo, has shown that we were ignorant of many of the salient features of the relief of the sea-bottom. Some of the features discovered by the Carnegie were as follows: "Carnegie Ridge," about 100 miles west of the coast of Ecuador, rises about 1800 meters above the general level of the ocean-floor in its vicinity; "Merriam Ridge," in about latitude 25°S. and longitude 82°W., is about 10 miles across and 3000 meters higher than the surrounding ocean-bed; "Bauer Deep" is in latitude about 15°S. and longitude 98°W., with a maximum depth of about 4500 meters; a new ridge 2000 meters above the general contour is in latitude 17.7°S. and longitude 141.6°W.; "Fleming Deep," in latitude 23.8°N. and longitude 144.1°E., has a maximum depth of 8650 meters, and is the sixth greatest depth discovered to date; and "Hayes Peak," in latitude 32.4°N. and longitude 127.8°W., rises about 3000 meters above the general level.

There are possibilities of wide distribution of marine organisms within the tropics and subtropics of the Pacific as conditions are now known to be, and it is possible that the number of shoals that could serve as landing places for migrating organisms is greater than the number now definitely known. Furthermore, as there are many ridges on the floor of the ocean, the number of shoals might be increased by moderate elevation of some of those areas. Therefore, to account for the distribution of the organisms here considered, orbitoid foraminifera and shoal-water corals, it is not necessary to postulate the former existence of extensive land areas or land bridges over those parts of the ocean now occupied by water of abyssal depths.

Conclusions.—1. It seems that the facts of the known geographic distribution of the orbitoid foraminifera, during successive geological epochs, can be accounted for by the transport of the organisms during larval stages by ocean currents, and that the postulation of the former

existence of extensive land areas where there are now oceanic waters of abyssal depths is not necessary. The maximum required change in the depths of the ocean would be that some of the suboceanic peaks and ridges should have stood near sea level, but it is not certain that even such changes were necessary.

2. Orbitoids during Upper Cretaceous time migrated between Europe and India by way of Tethys and between Europe and America across the Atlantic, but not from the east to the west side of America.

3. During Eocene time there was migration along Tethys, across the Atlantic, and from the east to the west side of America.

4. During Oligocene and most of Miocene time there was migration across the Central American region between the Atlantic and Pacific, and from the Pacific side of America to the Pacific Islands, probably as far as the East Indian region. There was also migration across the Atlantic between Europe and America. It appears that there may have been migration around the southern end of India from Europe to the East Indian region during Oligocene time. Migration around the south end of Africa, it seems, was impracticable because of low temperature and adverse currents.

5. Subsequent to Miocene the Pacific has been separated from the Atlantic both in the Central American and the Asiatic region.

6. It must be recognized that there are in the conclusions expressed above two postulates on which additional research is needed. The first of these concerns the paleogeography of that part of eastern Asia between the western end of the Himalaya Mountains and the eastern end of the Mediterranean, and thence southward to the western part of the Indian Ocean. Doubt regarding this matter is expressed in the last sentence of paragraph 4 (above) of these conclusions. The second postulate concerns the duration of the planketonic larval stages of the orbitoids. A satisfactory solution of this problem can be attained by a study of the reproductive cycles in those living genera of larger foraminifera of which fossil representatives occur in association with orbitoids. A list of genera appropriate for such a study is given on a preceding page.

7. This paper is not conclusive. It presents in summary form the more significant known facts, it proposes an interpretation which is to some degree hypothetical, and it points out two lines of research which are needed for a more definite solution of the problem.

In the following list of literature references to publications which may be found cited under the synonymy of the different genera of the orbitoids in Cushmans' "Foraminifera, their classification and economic use," second edition, are not given. Only those publications which are mentioned in the foregoing text are listed.

Astre, G., "Sur *Monolepidorbis*, foraminifère voisin des lindérines et des orbitoïdes," *Soc. géol. de France Bull.*, sér. 4, t. 27, 387–94, pl. 20 (1927).

Checchia-Rispoli, G., "Considerazioni paleontologiche sull'Eocene dei intorni di Castelluccio Valmaggiore (Capitanta)," *R. Accad. Naz. Lincei, Fisiche, Matematiche è Naturali, Mem.*, Anno 322, ser. sesta, **1**, fasc. 1, 1–13, 7 pls. (1925).

Cushman, J. A., *Foraminifera, Their Classification and Economic Use*," First edition, 401 pp., 59 pls. (1928), 2nd edition (1933).

Edmondson, C. H., "The Ecology of an Hawaiian Coral Reef," *Bernice P. Bishop Mus. Bull.*, **45**, 64 pp. (1928).

Edmondson, C. H., "Growth of Hawaiian Corals," *Bernice P. Bishop Mus. Bull.*, **58**, 38 pp. (1929).

Galloway, J. J., *A Manual of Foraminifera*, 483 pp., 42 pls. (1933).

Gregory, J. W., "The Geological History of the Atlantic Ocean," *Geol. Soc. London, Quart. Jour.*, **85**, pt. 1, lxviii–cxxii, Feb. 15 (1929).

Gregory, J. W., "The Geological History of the Pacific Ocean," *Geol. Soc. London, Quart. Jour.*, **86**, pt. 2, lxxii–cxxxvi, July 31 (1930).

Joleaud, L., "L'histoire paléogéographique de l'Océan Pacifique," *C. R. des Séances de la Soc. de Biogéog.*, no. **62**, 7th yr., 9–12 (1931).

Joleaud, L., "Les récents progrès de nos connaissances sur l'histoire du Pacifique aux temps tertiaires et la théorie de Wegener," *C. R. des Séances de l'Acad. Sci.*, t. 192, 628–29 (1931).

Lister, J. J., "The Foraminifera," in *Treatise on Zoölogy* edited by E. Ray Lancaster, pt. 1, 47–149 (1903).

Lister, J. J., "Life History of the Foraminifera," *Brit. Assoc. Adv. Sci.*, Report for 1906, Trans. Sect. D, 583–96 (1907).

Marshall, P., "Geology of Mangaia," *Bernice P. Bishop Mus. Bull.*, **36**, 48, 3 pls., 3 text-figs. (1927).

Martin, K., "Wann löste sich das Gebiet des indischen Archipels von der Tethys," *Samml. geol. Reichs-Mus. Leiden*, ser. 1, **9**, 337–55 (1914).

Martin, K., "Unsere palaeozoologische Kenntnis von Java, mit einleitenden Bemerkungen über die Geologie der Insel," *Samml. geolog. Reichs-Mus. Leiden, Beilage-Bd.*, 7 + 158, 4 pls. (1919).

Martin, K., "The Age of the Tertiary Sediments of Java," *Bernice P. Bishop Mus. Spec. Pub.* no. **7**, 754–65 (1921) (translated by T. W. Vaughan).

Myers, E. H. [Cultures of species of foraminifera.] In Vaughan, Oceanographic research at the Scripps Institution of Oceanography, University of California, April, 1932 to April, 1933. *Trans. Amer. Geophys. Union, 14th Ann. Meet., Nat. Res. Counc.*, 214 (1933).

Nuttall, W. L. F., "The Stratigraphy and Foraminifera of the Laki Series," *Quart. Jour. Geol. Soc.*, **81**, pt. 3, 417–53, pls. 23–27 (1925).

Pilgrim, Guy E., "The Tertiary Formations of India, and the Interrelation of the Marine and Terrestrial Deposits," *Second Pan-Pac. Sci. Cong., Australia*, 1923, **1**, 896–933 (1926).

Schaudinn, Fritz, "Untersuchungen über die Fortpflanzung einiger Rhizopoden," *Arb. a. d. Kais. Gesundheitsamte*, Bd. **19**, Heft 3, 547–76 (1903).

Schott, G., *Geographie des Atlantischen Ozeans*, 330 pp., 1 illustr., 28 tables, 90 text-figs., Hamburg (1912).

Schott, G., and Schu, F., "Die Wärmeverteilung in den Tiefen des Stillen Ozeans," *Ann. der Hydr. u. Mar. Meteor.*, 26 pp., mit 18 Karten u. 4 Profilen auf 15 Tafeln (1910).

Silvestri, A., "Fossili miocenici nel territorio di Bivona (Agrigento)," *Riv. Ital. di Paleont.*, Anno 37, Fasc. 1–2, 29–36, pls. 4, 5 (1931).

Umbgrove, J. H. "Tertiary Sea-Connections between Europe and the Indo-Pacific Area," *Fourth Pac. Sci. Cong., Batavia, Bandoeng, Java*, 1929, Proc., 2*a*, 91–104 (1931). (Preprint, 1929.)

van der Vlerk, I. M., "The Genus Lepidocyclina in the Far East," *Eclog. geol. Helv.* **21,** no. 1, 182–211, 18 pls., 3 tables (1928).

Vaughan, T. W., "Recent Madreporaria of the Hawaiian Islands and Laysan," *U. S. Nat. Mus. Bull.*, no. **59,** 9 + 427, 1 fig. in text, 96 pls., July (1907).

Vaughan, T. W., "Recent Madreporaria of Southern Florida," *Carnegie Inst. Wash. Year Bk.* no. **9** (for 1910), 135–44, pl. 1 (1911).

Vaughan, T. W., "The Reef-Coral Fauna of Carrizo Creek, Imperial County, California, and Its Significance," *U. S. Geol. Surv.*, Prof. Paper 98-T, 355–95, pls. 92–102, 4 figs. (1917).

Vaughan, T. W., "The Temperature of the Florida Coral-Reef Tract," *Carnegie Inst. Wash.*, Publ. 213, 319–39, 3 text-figs. (1918).

Vaughan, T. W., "Some Shoal-Water Corals from Murray Island (Australia), Cocos-Keeling Islands and Fanning Island," *Carnegie Inst. Wash.*, Publ. 213, 49–234, pls. 20–93, 2 text-figs. (1918).

Vaughan, T. W., "Fossil Corals from Central America, Cuba and Porto Rico, with an Account of the American Tertiary, Pleistocene and Recent Coral Reefs," *U. S. Nat. Mus. Bull.*, **103,** 189–524, pls. 68–152, 22 text-figs. (1919).

Vaughan, T. W., "Correlation of the Tertiary Formations of Central America and the West Indies," *Bernice P. Bishop Mus., Spec. Publ.*, no. **7,** pt. 3, 819–44 (1921).

Vaughan, T. W., Richards, H. C., Martin, K., Smith, Warren D., Yabe, H., Brooks, A. H., Clark, B. L., Berry, E. W., "Correlations of Post-Cretaceous Formations in the Pacific Region," *Bernice P. Bishop Mus., Spec. Publ.* no. **7,** pt. 3, 713–873 (1921).

Vaughan, T. W., Matthew, W. D., Kellogg, R., Berry, E. W., Stefanini, G., Canu, F., and Bassler, R. S., Cooke, C. W., Gardner, Julia, Woodring, W. P., "Symposium on Correlation of the Tertiary Formations of Southeastern North America, Central America and the West Indies with the Tertiary Formations of Europe," *Geol. Soc. Amer. Bull.*, **35,** 677–886, Dec. 30 (1924).

Vaughan, T. W., Cooke, C. W., Clark, B. L., Stock, Chester, Chaney, R. W., Smith D., Matsumoto, H., Pilgrim, E. G., Pritchard, G. D., Marshall, P., Cotter, G. de P., Chapman, F., Yabe, H., and Aoki, R., Stanley, E. R., Benson, W. N., Morgan, P. G., Osborn, H. F., Matthew, W. D., Singleton, F. A., "Correlation of the Cainozoic Formations in the Pacific Region," *Second Pan-Pacific Sci. Cong., Australia*, 1923, Proc. **1,** pp. 860–1024 (1926).

von Ihering, H., "Land-Bridges across the Atlantic and Pacific Oceans during the Kainozoic Era," *Geol. Soc. London Quart. Journ.*, **87,** 376–91 (1931).